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ON HYPERELLIPTIC FUNCTIONS OF GENUS TWO.

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IN this paper I have attempted to establish some of the fundamental properties of the hyperelliptic functions of genus two, taking as a starting point the addition theorem.

I have, to a great extent, taken as a model Halphen's *Traité des Fonctions Elliptiques*, Part I., in which the theory of Weierstrass's elliptic functions is treated from the same point of view.

The sextic which comes under the square root I suppose to be so transformed that one of its roots is infinite and another zero, and that the product of the other four is unity.

If $R(x) = \Pi(x - a_s)$, ($s = 1, 2, 3, 4, 5, 6$), this may be effected by a transformation

$$y = \left(-\frac{R'(a_s)}{R'(a_1)} \right)^{\frac{1}{4}} \cdot \frac{x - a_1}{x - a_s}.$$

I.

§ 1. I take

$$\Theta \equiv \theta^6 + b\theta^4 + c\theta^3 + d\theta^2 + \theta,$$

then u and v are defined by

$$\left. \begin{aligned} u &= \int_x^\infty \frac{dx}{\sqrt{X}} - \int_y^\infty \frac{dy}{\sqrt{Y}} \\ v &= \int_x^\infty \frac{x dx}{\sqrt{X}} - \int_y^\infty \frac{y dy}{\sqrt{Y}} \end{aligned} \right\} \dots\dots\dots (1),$$

so that

$$\left. \begin{aligned} du &= \frac{dy}{\sqrt{Y}} - \frac{dx}{\sqrt{X}} \\ dv &= \frac{y dy}{\sqrt{Y}} - \frac{x dx}{\sqrt{X}} \end{aligned} \right\},$$

and

$$\left. \begin{aligned} (x-y) \frac{dx}{\sqrt{X}} &= y du - dv \\ (x-y) \frac{dy}{\sqrt{Y}} &= x du - dv \end{aligned} \right\}.$$

Further I write

$$\left. \begin{aligned} \beta(u, v) &= \beta = xy \\ \alpha(u, v) &= \alpha = x + y \end{aligned} \right\} \dots\dots\dots(2).$$

Since u and v are changed to $-u, -v$ by changing the sign of $\sqrt{X}$ and $\sqrt{Y}$, α and β are even functions of u and v .

Differentiating, I get

$$\left. \begin{aligned} d\alpha &= dx + dy = \frac{y \sqrt{X} + x \sqrt{Y}}{x-y} du - \frac{\sqrt{X} + \sqrt{Y}}{x-y} dv \\ d\beta &= y dx + x dy = \frac{y^2 \sqrt{X} + x^2 \sqrt{Y}}{x-y} du - \frac{y \sqrt{X} + x \sqrt{Y}}{x-y} dv \end{aligned} \right\} \dots\dots\dots(3).$$

or

$$\left. \begin{aligned} \frac{\partial \alpha}{\partial v} &= -\frac{\sqrt{X} + \sqrt{Y}}{x-y}, \quad \frac{\partial \beta}{\partial u} = \frac{y^2 \sqrt{X} + x^2 \sqrt{Y}}{x-y} \\ \frac{\partial \alpha}{\partial u} &= -\frac{\partial \beta}{\partial v} = \frac{y \sqrt{X} + x \sqrt{Y}}{x-y} \end{aligned} \right\} \dots(4).$$

It follows at once that $-\alpha dv + \beta du$ is a perfect differential, and if I determine γ from the equation $\frac{\partial \gamma}{\partial v} = -\frac{\partial \beta}{\partial u}$, it will follow that

$$\beta dv - \gamma du \text{ and } -\alpha dv^2 + 2\beta du dv - \gamma du^2$$

are also perfect differentials.

§ 2. To determine γ . From

$$\frac{\partial \alpha}{\partial v} = - \frac{\sqrt{X} + \sqrt{Y}}{x - y}$$

I get

$$\begin{aligned} \frac{\partial^2 \alpha}{\partial v^2} &= \frac{\sqrt{X} + \sqrt{Y}}{(x - y)^2} \left(\frac{\partial x}{\partial v} - \frac{\partial y}{\partial v} \right) - \frac{1}{2} \frac{X'}{(x - y) \sqrt{X}} \frac{\partial x}{\partial v} \\ &\quad - \frac{1}{2} \frac{Y'}{(x - y) \sqrt{Y}} \frac{\partial y}{\partial v} \\ &= - \frac{\sqrt{X} + \sqrt{Y}}{(x - y)^2} \cdot \frac{\sqrt{X} - \sqrt{Y}}{x - y} + \frac{1}{2} \frac{X' + Y'}{(x - y)^2}, \end{aligned}$$

or
$$2(x - y)^2 \frac{\partial^2 \alpha}{\partial v^2} = X' + Y' - 2 \frac{X - Y}{x - y}.$$

But we have

$$\Theta \equiv \theta^5 + b\theta^4 + c\theta^3 + d\theta^2 + \theta,$$

and
$$\Theta' = 5\theta^4 + 4b\theta^3 + 3c\theta^2 + 2d\theta + 1.$$

And so

$$\begin{aligned} 2(x - y)^2 \frac{\partial^2 \alpha}{\partial v^2} &= 5(x^4 + y^4) + 4b(x^3 + y^3) + 3c(x^2 + y^2) + 2d(x + y) + 2 \\ &\quad - 2\{(x^4 + x^3y + x^2y^2 + xy^3 + y^4) + b(x^3 + x^2y + xy^2 + y^3) \\ &\quad + c(x^2 + xy + y^2) + d(x + y) + 1\} \\ &= 3x^4 - 2x^3y - 2x^2y^2 - 2xy^3 + 3y^4 \\ &\quad + 2b(x^3 - x^2y - xy^2 + y^3) + c(x^2 - 2xy + y^2) \\ &= (x - y)^2 \{c + 2b(x + y) + 3x^2 + 4xy + 3y^2\}, \end{aligned}$$

and therefore

$$2 \frac{\partial^2 \alpha}{\partial v^2} = c + 2b\alpha + 3\alpha^2 - 2\beta \dots\dots\dots (5).$$

Multiplying by $\frac{\partial \alpha}{\partial v}$, I get

$$2 \frac{\partial \alpha}{\partial v} \cdot \frac{\partial^2 \alpha}{\partial v^2} = c \frac{\partial \alpha}{\partial v} + 2b\alpha \frac{\partial \alpha}{\partial v} + 3\alpha^2 \frac{\partial \alpha}{\partial v} - 2\beta \frac{\partial \alpha}{\partial v}.$$

But from equations (4), I get at once

$$\beta \frac{\partial \alpha}{\partial v} - \alpha \frac{\partial \beta}{\partial v} = \frac{y^2 \sqrt{X} + x^2 \sqrt{Y}}{x - y} = \frac{\partial \beta}{\partial u} = - \frac{\partial \gamma}{\partial v} \dots (6),$$

and so

$$-2\beta \frac{\partial \alpha}{\partial v} = -\beta \frac{\partial \alpha}{\partial v} - \alpha \frac{\partial \beta}{\partial v} + \frac{\partial \gamma}{\partial v},$$

and therefore I may put

$$2 \frac{\partial \alpha}{\partial v} \cdot \frac{\partial^2 \alpha}{\partial v^2} = c \frac{\partial \alpha}{\partial v} + 2b\alpha \frac{\partial \alpha}{\partial v} + 3\alpha^2 \frac{\partial \alpha}{\partial v} - \beta \frac{\partial \alpha}{\partial v} - \alpha \frac{\partial \beta}{\partial v} + \frac{\partial \gamma}{\partial v},$$

and then, integrating, I take

$$\gamma = \left(\frac{\partial \alpha}{\partial v} \right)^2 - d - c\alpha - b\alpha^2 - \alpha^3 + \alpha\beta \dots (7).$$

The constant d is added for the sake of symmetry in relations to be developed later.

In terms of x and y this becomes

$$\begin{aligned} \gamma &= \left(\frac{\sqrt{X} + \sqrt{Y}}{x - y} \right)^2 - d - c(x + y) - b(x + y)^2 - (x + y)^3 + xy(x + y) \\ &= \frac{2\sqrt{XY} + x + y + 2dxy + cxy(x + y) + 2bx^2y^2 + x^2y^3(x + y)}{(x - y)^2} \\ &\dots (7a). \end{aligned}$$

This equation when rationalised and reduced gives as the relation between α, β, γ

$$\begin{aligned} (\beta^2 - c\beta + 1)^2 - 2\alpha\gamma(\beta^2 + c\beta + 1) + \alpha^2\gamma^2 \\ - 4\beta(\alpha^2 + \gamma^2) - 4b\beta(\alpha + \beta\gamma) - 4d\beta(\gamma + \alpha\beta) - 4bd\beta^3 = 0, \end{aligned}$$

which may also be written

$$(\alpha\gamma + 1 - c\beta + \beta^2)^2 = 4(\alpha + \beta\gamma + d\beta)(\gamma + \alpha\beta + b\beta) \dots (8).$$

§ 3. I proceed to find other relations between α, β, γ and their differential coefficients.

I have

$$\begin{aligned} \left(\frac{\partial\beta}{\partial v}\right)^2 - \beta \left(\frac{\partial\alpha}{\partial v}\right)^2 &= \left(\frac{y\sqrt{X} + x\sqrt{Y}}{x-y}\right)^2 - xy \left(\frac{\sqrt{X} + \sqrt{Y}}{x-y}\right)^2 \\ &= \frac{y^2X + x^2Y - xy(X+Y)}{(x-y)^2} = \frac{xy - yX}{x-y} \\ &= -xy(x^2 + x^2y + xy^2 + y^2) - bxy(x^2 + xy + y^2) \\ &\quad - cry(x+y) - dxy \\ &= -\beta\{\alpha^2 - 2\alpha\beta + b(\alpha^2 - \beta) + c\alpha + d\}, \end{aligned}$$

and therefore

$$\left(\frac{\partial\beta}{\partial v}\right)^2 = \beta(\gamma + \alpha\beta + b\beta) \dots\dots\dots(9).$$

Similarly

$$\begin{aligned} \left(\frac{\partial\beta}{\partial u}\right)^2 - \beta^2 \left(\frac{\partial\alpha}{\partial v}\right)^2 &= \left(\frac{y^2\sqrt{X} + x^2\sqrt{Y}}{x-y}\right)^2 - x^2y^2 \left(\frac{\sqrt{X} + \sqrt{Y}}{x-y}\right)^2 \\ &= (x+y) \cdot \frac{x^2Y - y^2X}{x-y} \\ &= \alpha\{-x^2y^2(x^2 + xy + y^2) - bx^2y^2(x+y) - cx^2y^2 + xy\} \\ &= -\alpha\beta^2(\alpha^2 - \beta) - b\alpha^2\beta^2 - c\alpha\beta^2 + \alpha\beta, \end{aligned}$$

and therefore

$$\left(\frac{\partial\beta}{\partial u}\right)^2 = \beta(\alpha + \gamma\beta + d\beta) \dots\dots\dots(10),$$

From these two relations we can obtain at once rational algebraical expressions for α and γ in terms of β , $\frac{\partial\beta}{\partial u}$, $\frac{\partial\beta}{\partial v}$, viz.

$$(\beta^2 - \beta)\alpha = \beta \left(\frac{\partial\beta}{\partial v}\right)^2 - \left(\frac{\partial\beta}{\partial u}\right)^2 + d\beta^2 - b\beta^2 \dots\dots(11),$$

$$(\beta^2 - \beta)\gamma = \beta \left(\frac{\partial\beta}{\partial u}\right)^2 - \left(\frac{\partial\beta}{\partial v}\right)^2 + b\beta^2 - d\beta^2 \dots\dots(12).$$

The symmetry of these relations in α and γ , u and v suggests the existence of relations for γ corresponding to (5) and (6), and these I proceed to deduce.

§ 4. From (9) I have at once

$$\frac{\gamma}{\beta} + \alpha + b = \frac{1}{\beta^2} \left(\frac{\partial \beta}{\partial v} \right)^2 = \frac{1}{(x-y)^2} \left\{ \frac{\sqrt{X}}{x} + \frac{\sqrt{Y}}{y} \right\},$$

and therefore

$$\begin{aligned} & \frac{\partial}{\partial u} \left(\frac{\gamma}{\beta} \right) + \frac{\partial \alpha}{\partial u} \\ &= \frac{2}{x-y} \left(\frac{\sqrt{X}}{x} + \frac{\sqrt{Y}}{y} \right) \left[-\frac{y\sqrt{X} + x\sqrt{Y}}{xy} \frac{\frac{\partial x}{\partial u} - \frac{\partial y}{\partial u}}{(x-y)^2} + \right. \\ & \quad \left. \frac{\frac{\partial x}{\partial u}}{x-y} \left(\frac{1}{2} \frac{X'}{x\sqrt{X}} - \frac{\sqrt{X}}{x^2} \right) + \frac{\frac{\partial y}{\partial u}}{x-y} \left(\frac{1}{2} \frac{Y'}{y\sqrt{Y}} - \frac{\sqrt{Y}}{y^2} \right) \right] \\ &= \frac{2}{\beta^2} \frac{\partial \alpha}{\partial u} \left[\frac{x^2 Y - y^2 X}{xy(x-y)^2} + \frac{1}{2} \frac{(xX' - 2X)y^2 + (yY' - 2Y)x^2}{x^2 y^2 (x-y)^2} \right] \\ &= \frac{1}{\beta^2} \frac{\partial \alpha}{\partial u} \frac{1}{(x-y)^2} \\ & \quad \times \left[-2x^2 y^2 (x^2 + xy + y^2) - 2bx^2 y^2 (x+y) - 2cx^2 y^2 + 2xy \right. \\ & \quad \left. + 3x^2 y^2 (x^2 + y^2) + 2bx^2 y^2 (x+y) + 2cx^2 y^2 - x^3 - y^3 \right] \\ &= \frac{1}{\beta^2} \frac{\partial \alpha}{\partial u} (x^2 y^2 - 1) = \frac{\partial \alpha}{\partial u} - \frac{1}{\beta^2} \frac{\partial \alpha}{\partial u}, \end{aligned}$$

and so

$$\frac{\partial}{\partial u} \left(\frac{\gamma}{\beta} \right) = -\frac{1}{\beta^2} \frac{\partial \alpha}{\partial u} = \frac{1}{\beta^2} \frac{\partial \beta}{\partial v},$$

$$\text{or} \quad \beta \frac{\partial \gamma}{\partial u} - \gamma \frac{\partial \beta}{\partial u} = \frac{\partial \beta}{\partial v} \dots \dots \dots (13).$$

From (13) I get

$$\beta^2 \left(\frac{\partial \gamma}{\partial u} \right)^2 = \gamma^2 \left(\frac{\partial \beta}{\partial u} \right)^2 + \left(\frac{\partial \beta}{\partial v} \right)^2 + 2\gamma \frac{\partial \beta}{\partial u} \cdot \frac{\partial \beta}{\partial v},$$

and from (6)

$$\beta^2 \left(\frac{\partial \alpha}{\partial v} \right)^2 = \alpha^2 \left(\frac{\partial \beta}{\partial v} \right)^2 + \left(\frac{\partial \beta}{\partial u} \right)^2 + 2\alpha \frac{\partial \beta}{\partial u} \cdot \frac{\partial \beta}{\partial v},$$

and therefore

$$\begin{aligned} \alpha\beta^2 \left(\frac{\partial\gamma}{\partial u} \right)^2 &= \alpha\gamma^2 \left(\frac{\partial\beta}{\partial u} \right)^2 + \alpha \left(\frac{\partial\beta}{\partial v} \right)^2 + \beta^2\gamma \left(\frac{\partial\alpha}{\partial v} \right)^2 - \alpha^2\gamma \left(\frac{\partial\beta}{\partial v} \right)^2 - \gamma \left(\frac{\partial\beta}{\partial u} \right)^2 \\ &= (\alpha\gamma^2 - \gamma) (\beta^2\gamma + \alpha\beta + d\beta^2) + (\alpha - \alpha^2\gamma) (\beta\gamma + \beta^2\alpha + b\beta^2) \\ &\quad + \beta^2\gamma (\gamma + d + c\alpha + b\alpha^2 + \alpha^2 - \alpha\beta) \\ &= \alpha\beta^2 \{ \gamma^2 + d\gamma^2 + c\gamma - \beta\gamma + \alpha + b \} \end{aligned}$$

or
$$\alpha = \left(\frac{\partial\gamma}{\partial u} \right)^2 - b - c\gamma - d\gamma^2 - \gamma^2 + \beta\gamma.$$

Differentiating this I get

$$\begin{aligned} 2 \frac{\partial\gamma}{\partial u} \cdot \frac{\partial^2\gamma}{\partial u^2} &= (3\gamma^2 + 2d\gamma + c - \beta) \frac{\partial\gamma}{\partial u} + \frac{\partial\alpha}{\partial u} - \gamma \frac{\partial\beta}{\partial u} \\ &= (3\gamma^2 + 2d\gamma + c - 2\beta) \frac{\partial\gamma}{\partial u}, \end{aligned}$$

or
$$2 \frac{\partial^2\gamma}{\partial u^2} = 3\gamma^2 + 2d\gamma + c - 2\beta \dots\dots\dots(14).$$

§ 5. Further, differentiating

$$\left(\frac{\partial\beta}{\partial u} \right)^2 = \beta^2\gamma + d\beta^2 + \alpha\beta \dots\dots\dots(10),$$

I get

$$2 \frac{\partial\beta}{\partial u} \cdot \frac{\partial^2\beta}{\partial u^2} = (2\beta\gamma + 2d\beta + \alpha) \frac{\partial\beta}{\partial u} + \beta^2 \frac{\partial\gamma}{\partial u} + \beta \frac{\partial\alpha}{\partial u}$$

from which

$$2 \frac{\partial^2\beta}{\partial u^2} = 3\beta\gamma + 2d\beta + \alpha \dots\dots\dots(15).$$

Similarly, from

$$\left(\frac{\partial\beta}{\partial v} \right)^2 = \beta^2\alpha + b\beta^2 + \gamma\beta \dots\dots\dots(9),$$

$$2 \frac{\partial\beta}{\partial v} \cdot \frac{\partial^2\beta}{\partial v^2} = (2\beta\alpha + 2b\beta + \gamma) \frac{\partial\beta}{\partial v} + \beta^2 \frac{\partial\gamma}{\partial v} + \beta^2 \frac{\partial\alpha}{\partial v},$$

and
$$2 \frac{\partial^2\beta}{\partial v^2} = 3\beta\alpha + 2b\beta + \gamma \dots\dots\dots(16).$$

From
$$\frac{\partial \beta}{\partial u} = \beta \frac{\partial \alpha}{\partial v} - \alpha \frac{\partial \beta}{\partial v},$$

I get

$$\frac{\partial^2 \beta}{\partial u \partial v} = \beta \frac{\partial^2 \alpha}{\partial v^2} - \alpha \frac{\partial^2 \beta}{\partial v^2},$$

or
$$2 \frac{\partial^2 \beta}{\partial u \partial v} = c\beta - \alpha\gamma - 2\beta^2 \dots\dots\dots(17).$$

But differentiating (9) with respect to u and (10) with respect to v , I get

$$\begin{aligned} 2 \frac{\partial \beta}{\partial v} \cdot \frac{\partial^2 \beta}{\partial u \partial v} &= (\gamma + 2\alpha\beta + 2b\beta) \frac{\partial \beta}{\partial u} + \beta \frac{\partial \gamma}{\partial u} + \beta^2 \frac{\partial \alpha}{\partial u} \\ &= 2(\gamma + \alpha\beta + b\beta) \frac{\partial \beta}{\partial u} + (1 - \beta^2) \frac{\partial \beta}{\partial v}, \end{aligned}$$

$$\begin{aligned} 2 \frac{\partial \beta}{\partial u} \cdot \frac{\partial^2 \beta}{\partial u \partial v} &= (\alpha + 2\beta\gamma + 2d\beta) \frac{\partial \beta}{\partial v} + \beta \frac{\partial \alpha}{\partial v} + \beta^2 \frac{\partial \gamma}{\partial v} \\ &= 2(\alpha + \beta\gamma + d\beta) \frac{\partial \beta}{\partial v} + (1 - \beta^2) \frac{\partial \beta}{\partial u}, \end{aligned}$$

and therefore, substituting from (17) in these last two equations,

$$\frac{\frac{\partial \beta}{\partial v}}{\frac{\partial \beta}{\partial u}} = \frac{2(\gamma + \alpha\beta + b\beta)}{c\beta - \alpha\gamma - 1 - \beta^2} = \frac{c\beta - \alpha\gamma - 1 - \beta^2}{2(\alpha + \beta\gamma + d\beta)},$$

verifying the relation between α, β, γ already found.

§ 6. It follows immediately from the relations already given, that

$$\left. \begin{aligned} \frac{\partial^2}{\partial u^2} (\log \beta) &= \frac{1}{\beta} \frac{\partial^2 \beta}{\partial u^2} - \frac{1}{\beta^2} \left(\frac{\partial \beta}{\partial u} \right)^2 = \frac{1}{2} \left(\gamma - \frac{\alpha}{\beta} \right) \\ \frac{\partial^2}{\partial u \partial v} (\log \beta) &= \frac{1}{\beta} \frac{\partial^2 \beta}{\partial u \partial v} - \frac{1}{\beta^2} \frac{\partial \beta}{\partial u} \cdot \frac{\partial \beta}{\partial v} = \frac{1}{2} \left(\frac{1}{\beta} - \beta \right) \\ \frac{\partial^2}{\partial v^2} (\log \beta) &= \frac{1}{\beta} \frac{\partial^2 \beta}{\partial v^2} - \frac{1}{\beta^2} \left(\frac{\partial \beta}{\partial v} \right)^2 = \frac{1}{2} \left(\alpha - \frac{\gamma}{\beta} \right) \end{aligned} \right\} \dots(18).$$

For convenience I collect here the principal formulæ already found:—

$$(\alpha\gamma + 1 - c\beta + \beta^2)^2 = 4(\alpha + \beta\gamma + d\beta)(\gamma + \alpha\beta + b\beta) \quad (\text{i});$$

$$\frac{\partial\alpha}{\partial u} = -\frac{\partial\beta}{\partial v} \quad (\text{ii}); \quad \frac{\partial}{\partial v} \left(\frac{\alpha}{\beta} \right) = -\frac{\partial}{\partial u} \left(\frac{1}{\beta} \right) \quad (\text{iii});$$

$$\frac{\partial\gamma}{\partial v} = -\frac{\partial\beta}{\partial u} \quad (\text{iv}); \quad \frac{\partial}{\partial u} \left(\frac{\gamma}{\beta} \right) = -\frac{\partial}{\partial v} \left(\frac{1}{\beta} \right) \quad (\text{v});$$

$$\left(\frac{\partial\beta}{\partial u} \right)^2 = \beta^2\gamma + \alpha\beta + d\beta^2 \quad (\text{vi});$$

$$\left(\frac{\partial\beta}{\partial v} \right)^2 = \beta^2\alpha + \beta\gamma + b\beta^2 \quad (\text{vii});$$

$$\frac{\partial\beta}{\partial u} \cdot \frac{\partial\beta}{\partial v} = \frac{1}{2}(c\beta^2 - \alpha\beta\gamma - \beta - \beta^2) \quad (\text{viii});$$

$$\left(\frac{\partial\alpha}{\partial v} \right)^2 = \gamma + d + c\alpha + b\alpha^2 + \alpha^2 - \alpha\beta \quad (\text{ix});$$

$$\left(\frac{\partial\gamma}{\partial u} \right)^2 = \alpha + b + c\gamma + d\gamma^2 + \gamma^2 - \beta\gamma \quad (\text{x});$$

$$\frac{\frac{\partial\beta}{\partial v}}{\frac{\partial\beta}{\partial u}} = \frac{2(\gamma + \alpha\beta + b\beta)}{c\beta - \alpha\gamma - 1 - \beta^2} = \frac{c\beta - \alpha\gamma - 1 - \beta^2}{2(\alpha + \gamma\beta + d\beta)} \quad (\text{xi});$$

$$\frac{\partial^2\beta}{\partial u^2} = \frac{1}{2}(3\beta\gamma + 2d\beta + \alpha) \quad (\text{xii});$$

$$\frac{\partial^2\beta}{\partial v^2} = \frac{1}{2}(3\alpha\beta + 2b\beta + \gamma) \quad (\text{xiii});$$

$$\frac{\partial^2\beta}{\partial u\partial v} = \frac{1}{2}(c\beta - \alpha\gamma - 2\beta^2) \quad (\text{xiv});$$

$$\frac{\partial^2\alpha}{\partial v^2} = \frac{1}{2}(c + 2b\alpha + 3\alpha^2 - 2\beta) \quad (\text{xv});$$

$$\frac{\partial^2\gamma}{\partial u^2} = \frac{1}{2}(c + 2d\gamma + 3\gamma^2 - 2\beta) \quad (\text{xvi}).$$

The period 2ω .

§ 7. From the relations (i)–(v) given above, it is evident that we may write

$$\frac{\gamma}{\beta} \text{ for } \alpha, \quad \frac{1}{\beta} \text{ for } \beta, \quad \frac{\alpha}{\beta} \text{ for } \gamma,$$

and the equations are still satisfied, leaving ∂u and ∂v unchanged.

That is, I may put

$$\frac{\gamma}{\beta} = \alpha(u + \omega, v + \omega'), \quad \frac{1}{\beta} = \beta(u + \omega, v + \omega'), \quad \frac{\alpha}{\beta} = \gamma(u + \omega, v + \omega')$$

from which

$$\alpha(u + 2\omega, v + 2\omega') = \alpha, \quad \beta(u + 2\omega, v + 2\omega') = \beta, \\ \gamma(u + 2\omega, v + 2\omega') = \gamma,$$

and $2\omega, 2\omega'$ constitute a period for the functions α, β, γ .

This will be called the period 2ω .

Approximate values for u, v , &c. for large values of x and y .

§ 8. I have

$$X = ax^5 + bx^4 + cx^3 + dx^2 + x,$$

and therefore

$$\sqrt{X} = x^{\frac{5}{2}} \left\{ 1 + \frac{b}{x} + \frac{c}{x^2} + \frac{d}{x^3} + \frac{1}{x^4} \right\}^{\frac{1}{2}} \\ = x^{\frac{5}{2}} + \frac{1}{2}bx^{\frac{3}{2}} + \left(\frac{c}{2} - \frac{b^2}{8} \right)x^{\frac{1}{2}} + \dots$$

Similarly

$$\frac{1}{\sqrt{X}} = \frac{1}{x^{\frac{5}{2}}} - \frac{1}{2} \frac{b}{x^{\frac{3}{2}}} + \left(\frac{3b^2}{8} - \frac{c}{2} \right) \frac{1}{x^{\frac{1}{2}}} \dots$$

and so

$$u = \int_x^\infty \frac{dx}{\sqrt{X}} - \int_y^\infty \frac{dy}{\sqrt{Y}} \\ = \frac{2}{3} \left(\frac{1}{x^{\frac{3}{2}}} - \frac{1}{y^{\frac{3}{2}}} \right) - \frac{1}{5}b \left(\frac{1}{x^{\frac{1}{2}}} - \frac{1}{y^{\frac{1}{2}}} \right) + \frac{1}{7} \left(\frac{3}{4}b^2 - c \right) \left(\frac{1}{x^{\frac{1}{2}}} - \frac{1}{y^{\frac{1}{2}}} \right) - \dots (19),$$

$$v = \int_x^\infty \frac{x dx}{\sqrt{X}} - \int_y^\infty \frac{y dy}{\sqrt{Y}} \\ = 2 \left(\frac{1}{x^{\frac{1}{2}}} - \frac{1}{y^{\frac{1}{2}}} \right) - \frac{1}{3}b \left(\frac{1}{x^{\frac{1}{2}}} - \frac{1}{y^{\frac{1}{2}}} \right) + \frac{1}{5} \left(\frac{3}{4}b^2 - c \right) \left(\frac{1}{x^{\frac{1}{2}}} - \frac{1}{y^{\frac{1}{2}}} \right) + \dots (20).$$

Further, since $-\alpha dv + \beta du =$

$$\begin{aligned} & -(x+y) \left(\frac{y dy}{\sqrt{Y}} - \frac{x dx}{\sqrt{X}} \right) + xy \left(\frac{dy}{\sqrt{Y}} - \frac{dx}{\sqrt{X}} \right) \\ & = -y^2 \frac{dy}{\sqrt{Y}} + x^2 \frac{dx}{\sqrt{X}}, \end{aligned}$$

I get

$$\begin{aligned} \int_{x,y}^{\mu,v} -\alpha dv + \beta du &= \int_{x,y} -\frac{x^2 dx}{\sqrt{X}} + y^2 \frac{dy}{\sqrt{Y}} \\ &= 2(x^{\frac{3}{2}} - y^{\frac{3}{2}}) + b \left(\frac{1}{x^{\frac{1}{2}}} - \frac{1}{y^{\frac{1}{2}}} \right) - \frac{1}{2} \left(\frac{3}{2} b^2 - c \right) \left(\frac{1}{x^{\frac{3}{2}}} - \frac{1}{y^{\frac{3}{2}}} \right) + \dots (21), \end{aligned}$$

I have further from (7a)

$$\begin{aligned} (x-y)^2 \gamma &= 2 \sqrt{XY} + x + y + 2dxy + cxy(x+y) \\ &\quad + 2bx^2y^2 + x^2y^2(x+y) \\ &= 2x^{\frac{1}{2}}y^{\frac{1}{2}} \left\{ 1 + \frac{b}{2} \frac{x+y}{xy} + \left(\frac{c}{2} - \frac{b^2}{8} \right) \frac{x^2+y^2}{x^2y^2} + \frac{1}{4} \frac{b^2}{xy} + \dots \right\} \\ &\quad + x^2y^2(x+y) + 2bx^2y^2 + cxy(x+y) + \dots \\ &= x^2y^2(x^{\frac{1}{2}} + y^{\frac{1}{2}})^2 + bx^{\frac{3}{2}}y^{\frac{3}{2}}(x^{\frac{1}{2}} + y^{\frac{1}{2}})^2 - \frac{b^2}{4} x^{\frac{1}{2}}y^{\frac{1}{2}}(x-y)^2 \\ &\quad + cx^{\frac{3}{2}}y^{\frac{3}{2}}(x^{\frac{1}{2}} + y^{\frac{1}{2}})^2(x - x^{\frac{1}{2}}y^{\frac{1}{2}} + y) + \dots, \end{aligned}$$

and therefore

$$\begin{aligned} \gamma &= \frac{x^2y^2}{(x^{\frac{1}{2}} - y^{\frac{1}{2}})^2} + b \frac{x^{\frac{3}{2}}y^{\frac{3}{2}}}{(x^{\frac{1}{2}} - y^{\frac{1}{2}})^2} - \frac{b^2}{4} x^{\frac{1}{2}}y^{\frac{1}{2}} \\ &\quad + c \frac{x^{\frac{3}{2}}y^{\frac{3}{2}} \{x - x^{\frac{1}{2}}y^{\frac{1}{2}} + y\}}{(x^{\frac{1}{2}} - y^{\frac{1}{2}})^2} \dots\dots (22). \end{aligned}$$

Substituting this value of γ I get

$$\begin{aligned} & \int_{x,y}^{\mu,v} \beta dv - \gamma du \\ &= \int_{x,y} xy \left\{ \frac{dx}{x^{\frac{3}{2}}} - \frac{dy}{y^{\frac{3}{2}}} - \frac{b}{2} \left(\frac{dx}{x^{\frac{3}{2}}} - \frac{dy}{y^{\frac{3}{2}}} \right) + \frac{1}{2} \left(\frac{3}{2} b^2 - c \right) \left(\frac{dx}{x^{\frac{5}{2}}} - \frac{dy}{y^{\frac{5}{2}}} \right) + \dots \right\} \\ &\quad - \left\{ \frac{x^2y^2}{(x^{\frac{1}{2}} - y^{\frac{1}{2}})^2} + b \frac{x^{\frac{3}{2}}y^{\frac{3}{2}}}{(x^{\frac{1}{2}} - y^{\frac{1}{2}})^2} + \dots \right\} \left\{ \frac{dx}{x^{\frac{3}{2}}} - \frac{dy}{y^{\frac{3}{2}}} - \frac{b}{2} \left(\frac{dx}{x^{\frac{3}{2}}} - \frac{dy}{y^{\frac{3}{2}}} \right) + \dots \right\}. \end{aligned}$$

The term of the highest order under the sign of integration is

$$xy \left(\frac{dx}{x^{\frac{1}{2}}} - \frac{dy}{y^{\frac{1}{2}}} \right) - \frac{x^2 y^2}{(x^{\frac{1}{2}} - y^{\frac{1}{2}})^2} \left(\frac{dx}{x^{\frac{1}{2}}} - \frac{dy}{y^{\frac{1}{2}}} \right),$$

or
$$\frac{y dx}{x^{\frac{1}{2}} (x^{\frac{1}{2}} - y^{\frac{1}{2}})^2} (x - 2x^{\frac{1}{2}} y^{\frac{1}{2}}) - \frac{x dy}{y^{\frac{1}{2}} (x^{\frac{1}{2}} - y^{\frac{1}{2}})^2} (y - 2x^{\frac{1}{2}} y^{\frac{1}{2}}),$$

or
$$y dx \frac{(x^{\frac{1}{2}} - 2y^{\frac{1}{2}})}{(x^{\frac{1}{2}} - y^{\frac{1}{2}})^2} - x dy \frac{(y^{\frac{1}{2}} - 2x^{\frac{1}{2}})}{(x^{\frac{1}{2}} - y^{\frac{1}{2}})^2},$$

or
$$\frac{2(y dx + x dy)}{x^{\frac{1}{2}} - y^{\frac{1}{2}}} - \frac{xy}{(x^{\frac{1}{2}} - y^{\frac{1}{2}})^2} \left(\frac{dx}{x^{\frac{1}{2}}} - \frac{dy}{y^{\frac{1}{2}}} \right),$$

that is
$$2d \left(\frac{xy}{x^{\frac{1}{2}} - y^{\frac{1}{2}}} \right);$$

and further the coefficient of b is

$$-\frac{1}{2} xy \left(\frac{dx}{x^{\frac{1}{2}}} - \frac{dy}{y^{\frac{1}{2}}} \right) - \frac{x^{\frac{1}{2}} y^{\frac{1}{2}}}{(x^{\frac{1}{2}} - y^{\frac{1}{2}})^2} \left(\frac{dx}{x^{\frac{1}{2}}} - \frac{dy}{y^{\frac{1}{2}}} \right) + \frac{1}{2} \frac{x^2 y^2}{(x^{\frac{1}{2}} - y^{\frac{1}{2}})^2} \left(\frac{dx}{x^{\frac{1}{2}}} - \frac{dy}{y^{\frac{1}{2}}} \right),$$

or
$$-\frac{1}{2} \frac{xy}{(x^{\frac{1}{2}} - y^{\frac{1}{2}})^2} \left[\frac{dx}{x^{\frac{1}{2}}} (x + y - 2x^{\frac{1}{2}} y^{\frac{1}{2}} + 2x^{\frac{1}{2}} y^{\frac{1}{2}} - y) \right. \\ \left. - \frac{dy}{y^{\frac{1}{2}}} (x + y - 2x^{\frac{1}{2}} y^{\frac{1}{2}} + 2x^{\frac{1}{2}} y^{\frac{1}{2}} - x) \right],$$

that is
$$-\frac{1}{2} \frac{xy}{(x^{\frac{1}{2}} - y^{\frac{1}{2}})^2} \left\{ \frac{dx}{x^{\frac{1}{2}}} - \frac{dy}{y^{\frac{1}{2}}} \right\},$$

which equals

$$\frac{1}{2(x^{\frac{1}{2}} - y^{\frac{1}{2}})^2} \left\{ (x^{\frac{1}{2}} - y^{\frac{1}{2}}) \left(y^{\frac{1}{2}} \frac{dx}{x^{\frac{1}{2}}} + x^{\frac{1}{2}} \frac{dy}{y^{\frac{1}{2}}} \right) - x^{\frac{1}{2}} y^{\frac{1}{2}} \left(\frac{dx}{x^{\frac{1}{2}}} - \frac{dy}{y^{\frac{1}{2}}} \right) \right\},$$

that is
$$d \left(\frac{x^{\frac{1}{2}} y^{\frac{1}{2}}}{x^{\frac{1}{2}} - y^{\frac{1}{2}}} \right).$$

Therefore I have

$$\int^u \beta dv - \gamma du = -2 \frac{xy}{x^{\frac{1}{2}} - y^{\frac{1}{2}}} - b \frac{x^{\frac{1}{2}} y^{\frac{1}{2}}}{x^{\frac{1}{2}} - y^{\frac{1}{2}}} \dots \dots (23).$$

§ 9. From (21) and (23) I shall get

$$\iint^u, v - \alpha dv^2 + 2\beta du dv - \gamma du^2$$

$$= \int_{x, y} \left\{ 2(x^{\frac{1}{2}} - y^{\frac{1}{2}}) + b \left(\frac{1}{x^{\frac{1}{2}}} - \frac{1}{y^{\frac{1}{2}}} \right) \right\} \left\{ \frac{dx}{x^{\frac{1}{2}}} - \frac{dy}{y^{\frac{1}{2}}} - \frac{b}{2} \left(\frac{dx}{x^{\frac{1}{2}}} - \frac{dy}{y^{\frac{1}{2}}} \right) \dots \right\}$$

$$- \left\{ \frac{2xy}{x^{\frac{1}{2}} - y^{\frac{1}{2}}} + b \frac{x^{\frac{1}{2}}y^{\frac{1}{2}}}{x^{\frac{1}{2}} - y^{\frac{1}{2}}} \dots \right\} \left\{ \frac{dx}{x^{\frac{1}{2}}} - \frac{dy}{y^{\frac{1}{2}}} - \frac{b}{2} \left(\frac{dx}{x^{\frac{1}{2}}} - \frac{dy}{y^{\frac{1}{2}}} \right) \dots \right\} \dots (24).$$

The terms of the highest order under the integral sign are

$$2(x^{\frac{1}{2}} - y^{\frac{1}{2}}) \left(\frac{dx}{x^{\frac{1}{2}}} - \frac{dy}{y^{\frac{1}{2}}} \right) - 2 \frac{xy}{x^{\frac{1}{2}} - y^{\frac{1}{2}}} \left(\frac{dx}{x^{\frac{1}{2}}} - \frac{dy}{y^{\frac{1}{2}}} \right),$$

or

$$2 \frac{1}{x^{\frac{1}{2}} - y^{\frac{1}{2}}} \left[\frac{dx}{x^{\frac{1}{2}}} (x - 2x^{\frac{1}{2}}y^{\frac{1}{2}}) - \frac{dy}{y^{\frac{1}{2}}} (y - 2x^{\frac{1}{2}}y^{\frac{1}{2}}) \right],$$

that is

$$4 \frac{dx}{x} + 4 \frac{dy}{y} - \frac{2}{x^{\frac{1}{2}} - y^{\frac{1}{2}}} \left(\frac{dx}{x^{\frac{1}{2}}} - \frac{dy}{y^{\frac{1}{2}}} \right),$$

or

$$4d \left(\log \frac{xy}{x^{\frac{1}{2}} - y^{\frac{1}{2}}} \right).$$

The coefficient of b is

$$\left(\frac{1}{x^{\frac{1}{2}}} - \frac{1}{y^{\frac{1}{2}}} \right) \left(\frac{dx}{x^{\frac{1}{2}}} - \frac{dy}{y^{\frac{1}{2}}} \right) - (x^{\frac{1}{2}} - y^{\frac{1}{2}}) \left(\frac{dx}{x^{\frac{1}{2}}} - \frac{dy}{y^{\frac{1}{2}}} \right)$$

$$- \frac{x^{\frac{1}{2}}y^{\frac{1}{2}}}{x^{\frac{1}{2}} - y^{\frac{1}{2}}} \left(\frac{dx}{x^{\frac{1}{2}}} - \frac{dy}{y^{\frac{1}{2}}} \right) + \frac{xy}{x^{\frac{1}{2}} - y^{\frac{1}{2}}} \left(\frac{dx}{x^{\frac{1}{2}}} - \frac{dy}{y^{\frac{1}{2}}} \right),$$

that is

$$\frac{dx}{x^{\frac{1}{2}}y^{\frac{1}{2}}(x^{\frac{1}{2}} - y^{\frac{1}{2}})} \{ -x^{\frac{1}{2}}(x^{\frac{1}{2}} - y^{\frac{1}{2}})^2 - y^{\frac{1}{2}}(x^{\frac{1}{2}} - y^{\frac{1}{2}})^2 - x^{\frac{1}{2}}y + y^{\frac{1}{2}} \}$$

$$- \frac{dy}{x^{\frac{1}{2}}y^{\frac{1}{2}}(x^{\frac{1}{2}} - y^{\frac{1}{2}})} \{ -y^{\frac{1}{2}}(x^{\frac{1}{2}} - y^{\frac{1}{2}})^2 - x^{\frac{1}{2}}(x^{\frac{1}{2}} - y^{\frac{1}{2}})^2 - xy^{\frac{1}{2}} + x^{\frac{1}{2}} \},$$

or

$$- \frac{dx}{x^{\frac{1}{2}}y^{\frac{1}{2}}} - \frac{dy}{x^{\frac{1}{2}}y^{\frac{1}{2}}},$$

that is

$$2d \left(\frac{1}{x^{\frac{1}{2}}y^{\frac{1}{2}}} \right).$$

Therefore I get finally

$$\iint^u, v - \alpha dv^2 + 2\beta du dv - \gamma du^2 = -4 \log \frac{xy}{x^{\frac{1}{2}} - y^{\frac{1}{2}}} - \frac{2b}{x^{\frac{1}{2}}y^{\frac{1}{2}}} + \dots \dots (25).$$

The functions σ , Z , ξ .

§ 10. I define $\sigma(u, v)$ therefore by the equation

$$4 \log \sigma = -2 \log \gamma + \iint_{0,0} \left(2 \frac{\partial^2}{\partial v^2} (\log \gamma) - \alpha \right) dv^2 \\ + 2 \left(2 \frac{\partial^2}{\partial u \partial v} (\log \gamma) + \beta \right) du dv + \left(2 \frac{\partial^2}{\partial u^2} (\log \gamma) - \gamma \right) du^2 \dots (26),$$

which may also be written in the form

$$\log(\gamma^2 \sigma^4) = \iint_{0,0} -\frac{\beta}{\gamma^2} (2\alpha + 2d\beta + c\gamma) dv^2 + \frac{2}{\gamma^2} (c\beta - 1 - \beta^2) du dv \\ - \frac{1}{\gamma^2} (2\alpha + 2b + c\gamma) du^2 \\ = F(u, v), \text{ say, so that } \sigma = \gamma^{-\frac{1}{2}} e^{\frac{1}{2}F} \dots (27).$$

When u, v are small, that is when x, y are large, I get

$$\log(\gamma^2 \sigma^4) = -\frac{c}{3} \left(\frac{1}{x^{\frac{1}{2}}} - \frac{1}{y^{\frac{1}{2}}} \right)^4 \text{ approximately } \dots (28).$$

Then the definition of $Z(u, v)$ and of $\xi(u, v)$ will be

$$Z = \frac{\partial}{\partial v} (\log \sigma) \dots (29),$$

$$\text{and} \quad \xi = \frac{\partial}{\partial u} (\log \sigma) \dots (30).$$

When u, v are small I get

$$Z + \frac{1}{2\gamma} \frac{\partial \gamma}{\partial v} = -\frac{c}{6} \left(\frac{1}{x^{\frac{1}{2}}} - \frac{1}{y^{\frac{1}{2}}} \right)^3 \text{ approximately,}$$

$$\text{and} \quad \xi + \frac{1}{2\gamma} \frac{\partial \gamma}{\partial u} = -\frac{1}{10} \left(\frac{1}{x^{\frac{1}{2}}} - \frac{1}{y^{\frac{1}{2}}} \right)^5 \text{ approximately.}$$

Further I have

$$-\frac{1}{4}\alpha = \frac{\partial Z}{\partial v} = \frac{\partial^2}{\partial v^2} (\log \sigma) \dots (31),$$

$$\frac{1}{4}\beta = \frac{\partial Z}{\partial u} = \frac{\partial \xi}{\partial v} = \frac{\partial^2}{\partial u \partial v} (\log \sigma) \dots (32),$$

$$-\frac{1}{4}\gamma = \frac{\partial \xi}{\partial u} = \frac{\partial^2}{\partial u^2} (\log \sigma) \dots (33).$$

The parity of the functions.

§ 11. If I write mx for x , my for y , mb for b , m^2c for c , m^2d for d , and put $m^2=1$, then $\sqrt{X}$, $\sqrt{Y}$ become $\sqrt{m}\sqrt{X}$, $\sqrt{m}\sqrt{Y}$; u, v become $\sqrt{m}u, m\sqrt{m}v$; α, β, γ become $m\alpha, m^2\beta, m^2\gamma$; Z, ζ become $m^2\sqrt{m}Z, m^2\sqrt{m}\zeta$; $F(u, v)$ is unchanged; and σ therefore becomes $(\sqrt{m^2})^{-1}\sigma$.

Therefore, putting $\sqrt{m}=-1$, I have that

α, β, γ are even functions of u, v ,

Z, ζ, σ „ odd „ „ u, v .

Also, putting $m=-1$,

u, v become $iu, -iv$,

α, β, γ become $-\alpha, \beta, -\gamma$,

Z, ζ become $iZ, -i\zeta$,

σ becomes $-i\sigma$,

whilst

b, c, d become $-b, c, -d$.

The addition theorem.

§ 12. If we apply Abel's theorem to the intersections of the two curves

$$\eta^2 = \xi^5 + b\xi^4 + c\xi^3 + d\xi^2 + \xi \dots\dots\dots(34),$$

$$E\eta = A\xi^3 + B\xi^2 + C\xi + D \dots\dots\dots(35),$$

we get an integral of the equations

$$\Sigma_r \frac{d\xi_r}{\eta_r} = 0, \quad \Sigma_r \frac{\xi_r d\xi_r}{\eta_r} = 0; \quad (r=1, 2, 3, 4, 5, 6),$$

where ξ_r is a root of the equation

$$(A\xi^3 + B\xi^2 + C\xi + D)^2 - E^2(\xi^5 + b\xi^4 + c\xi^3 + d\xi^2 + \xi) = 0$$

.....(36)

in the form

$$E\eta_r = A\xi_r^3 + B\xi_r^2 + C\xi_r + D, \quad (r=1, 2, 3, 4, 5, 6).$$

Altering the notation, I take the roots of (36) to be $x_1, y_1, x_2, y_2, x_3, y_3$; and the corresponding values of η_r to be $\sqrt{X_1}$,

$-\sqrt{Y_1}, \sqrt{X_2}, -\sqrt{Y_2}, \sqrt{X_3}, -\sqrt{Y_3}$; and then I get the six relations

$$\left. \begin{aligned} Ax_r^3 + Bx_r^2 + Cx_r + D - E\sqrt{X_r} &= 0 \\ Ay_r^3 + By_r^2 + Cy_r + D + E\sqrt{Y_r} &= 0 \end{aligned} \right\} (r = 1, 2, 3) \dots (37),$$

as the equivalent of

$$\left. \begin{aligned} \Sigma_r \left[\int_{x_r}^{\infty} \frac{dx_r}{\sqrt{X_r}} - \int_{y_r}^{\infty} \frac{dy_r}{\sqrt{Y_r}} \right] &= 0 \\ \Sigma_r \left[\int_{x_r}^{\infty} \frac{x_r dx_r}{\sqrt{X_r}} - \int_{y_r}^{\infty} \frac{y_r dy_r}{\sqrt{Y_r}} \right] &= 0 \end{aligned} \right\} (r = 1, 2, 3) \dots (38),$$

it being allowable to take all the upper limits ∞ , since all the roots of the equation (36) are infinite if we take A, B, C, E all zero.

With the notation of the previous section this is

$$u_1 + u_2 + u_3 = 0, \quad v_1 + v_2 + v_3 = 0;$$

I proceed to transform equations (37).

$$\text{From} \quad Ax^3 + Bx^2 + Cx + D - E\sqrt{X} = 0,$$

$$Ay^3 + By^2 + Cy + D + E\sqrt{Y} = 0,$$

I get at once

$$Axy(x+y) + Bxy - D - E \frac{y\sqrt{X} + x\sqrt{Y}}{x-y} = 0,$$

$$Ax^2y^2 - Cxy - D(x+y) - E \frac{y^2\sqrt{X} + x^2\sqrt{Y}}{x-y} = 0,$$

which may be written

$$\left. \begin{aligned} A\alpha + B - D \frac{1}{\beta} + E \frac{1}{\beta} \frac{\partial \beta}{\partial v} &= 0 \\ A\beta - C - D \frac{\alpha}{\beta} - E \frac{1}{\beta} \frac{\partial \beta}{\partial u} &= 0 \end{aligned} \right\} \dots \dots \dots (38).$$

Then writing in the suffixes r and s , any two of the numbers 1, 2, 3, I get from (38), by subtraction,

$$\left. \begin{aligned} A(\alpha_r - \alpha_s) - D \left(\frac{1}{\beta_r} - \frac{1}{\beta_s} \right) + E \left\{ \frac{1}{\beta_r} \left(\frac{\partial \beta}{\partial v} \right)_r - \frac{1}{\beta_s} \left(\frac{\partial \beta}{\partial v} \right)_s \right\} &= 0 \\ A(\beta_r - \beta_s) - D \left(\frac{\alpha_r}{\beta_r} - \frac{\alpha_s}{\beta_s} \right) - E \left\{ \frac{1}{\beta_r} \left(\frac{\partial \beta}{\partial u} \right)_r - \frac{1}{\beta_s} \left(\frac{\partial \beta}{\partial u} \right)_s \right\} &= 0 \end{aligned} \right\} \dots \dots \dots (39),$$

so that I may take

$$\left. \begin{aligned} A &= (\beta_r - \beta_s) \left\{ \frac{1}{\beta_r} \left(\frac{\partial \beta}{\partial u} \right)_r - \frac{1}{\beta_s} \left(\frac{\partial \beta}{\partial u} \right)_s \right\} \\ &\quad - (\alpha_r \beta_s - \alpha_s \beta_r) \left\{ \frac{1}{\beta_r} \left(\frac{\partial \beta}{\partial v} \right)_r - \frac{1}{\beta_s} \left(\frac{\partial \beta}{\partial v} \right)_s \right\} \\ -D &= \beta_r \beta_s \left[(\alpha_r - \alpha_s) \left\{ \frac{1}{\beta_r} \left(\frac{\partial \beta}{\partial u} \right)_r - \frac{1}{\beta_s} \left(\frac{\partial \beta}{\partial u} \right)_s \right\} \right. \\ &\quad \left. + (\beta_r - \beta_s) \left\{ \frac{1}{\beta_r} \left(\frac{\partial \beta}{\partial v} \right)_r - \frac{1}{\beta_s} \left(\frac{\partial \beta}{\partial v} \right)_s \right\} \right] \\ E &= (\beta_r - \beta_s)^2 + (\alpha_r - \alpha_s) (\alpha_r \beta_s - \alpha_s \beta_r) \end{aligned} \right\} \dots\dots\dots(40),$$

and one form of the addition theorem is that the ratios $A:D:E$ are unaltered, whatever two of the numbers 1, 2, 3 are written for r and s .

I have also from (38)

$$-2B = A (\alpha_1 + \alpha_2) - D \left(\frac{1}{\beta_1} + \frac{1}{\beta_2} \right) + E \left\{ \frac{1}{\beta_1} \left(\frac{\partial \beta}{\partial v} \right)_1 + \frac{1}{\beta_2} \left(\frac{\partial \beta}{\partial v} \right)_2 \right\} \dots\dots\dots(41),$$

and

$$-2C = D \left(\frac{\alpha_1}{\beta_1} + \frac{\alpha_2}{\beta_2} \right) - A (\beta_1 + \beta_2) + E \left\{ \frac{1}{\beta_1} \left(\frac{\partial \beta}{\partial u} \right)_1 + \frac{1}{\beta_2} \left(\frac{\partial \beta}{\partial u} \right)_2 \right\} \dots\dots\dots(42).$$

The following symmetrical expressions are also to be noticed, viz.

$$\frac{E}{A} = \frac{-\Sigma \alpha_i \beta_j (\beta_i - \beta_j)}{\Sigma \left(\frac{\partial \beta}{\partial v} \right)_i (\beta_i - \beta_j)} = \frac{\Sigma \alpha_i \beta_j \beta_k (\beta_i - \beta_j)}{\Sigma \left(\frac{\partial \beta}{\partial u} \right)_i (\alpha_i \beta_j - \alpha_j \beta_i)},$$

$$\frac{E}{D} = \frac{\Sigma \alpha_i \beta_j (\beta_i - \beta_j)}{\Sigma \left(\frac{\partial \beta}{\partial v} \right)_i \beta_j \beta_k (\alpha_i - \alpha_j)} = \frac{-\Sigma \alpha_i \beta_j \beta_k (\beta_i - \beta_j)}{\Sigma \left(\frac{\partial \beta}{\partial u} \right)_i \beta_j \beta_k (\beta_j - \beta_k)},$$

which are got at once by solving the systems of equations (38) separately.

§ 13. The relation (36) may, however, be treated in another way.

As an equivalent of the equations (37), we may say that we have the identity

$$(A\theta^3 + B\theta^2 + C\theta + D)^3 - E^3(\theta^3 + b\theta^2 + c\theta^2 + d\theta^3 + \theta) \\ \equiv A^3(\theta - x_1)(\theta - y_1)(\theta - x_2)(\theta - y_2)(\theta - x_3)(\theta - y_3) \dots (43),$$

and by equating coefficients in this identity, I get

$$x_1 + y_1 + x_2 + y_2 + x_3 + y_3 = \frac{E^3}{A^3} - \frac{2B}{A} \dots (44)$$

$$\frac{1}{x_1} + \frac{1}{y_1} + \frac{1}{x_2} + \frac{1}{y_2} + \frac{1}{x_3} + \frac{1}{y_3} = \frac{E^3}{D^3} - \frac{2C}{D} \dots (45),$$

$$x_1 y_1 x_2 y_2 x_3 y_3 = \frac{D^3}{A^3} \dots (46),$$

which, in the other notation, are written

$$\alpha_1 + \alpha_2 + \alpha_3 = \frac{E^3}{A^3} - \frac{2B}{A} \dots (47),$$

$$\frac{\alpha_1}{\beta_1} + \frac{\alpha_2}{\beta_2} + \frac{\alpha_3}{\beta_3} = \frac{E^3}{D^3} - \frac{2C}{D} \dots (48),$$

$$\beta_1 \beta_2 \beta_3 = \frac{D^3}{A^3} \dots (49).$$

By these formulæ $\alpha(u_1 + u_2, v_1 + v_2)$ and $\beta(u_1 + u_2, v_1 + v_2)$ are expressed explicitly in terms of the functions of (u_1, v_1) and (u_2, v_2) .

§ 14. I must now investigate several transformations and identities which are necessary for the further development.

Writing

$$\Phi \equiv \gamma_1 - \gamma_2 + \alpha_1 \beta_2 - \alpha_2 \beta_1 \dots (50),$$

$$\Phi' \equiv \alpha_1 - \alpha_2 + \gamma_1 \beta_2 - \gamma_2 \beta_1 \dots (51),$$

$$\Delta \equiv \alpha_1 \gamma_1 \beta_2 - \alpha_2 \gamma_2 \beta_1 + (\beta_1 - \beta_2)(1 - \beta_1 \beta_2) \dots (52),$$

I get

$$\begin{aligned}
 A &= (\beta_1 - \beta_2) \left\{ \frac{1}{\beta_1} \left(\frac{\partial \beta}{\partial u} \right)_1 - \frac{1}{\beta_2} \left(\frac{\partial \beta}{\partial u} \right)_2 \right\} \\
 &\quad - (\alpha_1 \beta_2 - \alpha_2 \beta_1) \left\{ \frac{1}{\beta_1} \left(\frac{\partial \beta}{\partial v} \right)_1 - \frac{1}{\beta_2} \left(\frac{\partial \beta}{\partial v} \right)_2 \right\} \\
 &= \left(\frac{\partial \beta}{\partial u} \right)_1 + \left(\frac{\partial \beta}{\partial u} \right)_2 + \alpha_2 \left(\frac{\partial \beta}{\partial v} \right)_1 + \alpha_1 \left(\frac{\partial \beta}{\partial v} \right)_2 \\
 &\quad - \beta_2 \left\{ \frac{1}{\beta_1} \left(\frac{\partial \beta}{\partial v} \right)_1 + \frac{\alpha_1}{\beta_1} \left(\frac{\partial \beta}{\partial v} \right)_2 \right\} - \beta_1 \left\{ \frac{1}{\beta_2} \left(\frac{\partial \beta}{\partial v} \right)_1 + \frac{\alpha_2}{\beta_2} \left(\frac{\partial \beta}{\partial v} \right)_2 \right\} \\
 &= - \left(\frac{\partial \gamma}{\partial v} \right)_1 - \left(\frac{\partial \gamma}{\partial v} \right)_2 + \alpha_2 \left(\frac{\partial \beta}{\partial v} \right)_1 + \alpha_1 \left(\frac{\partial \beta}{\partial v} \right)_2 - \beta_2 \left(\frac{\partial \alpha}{\partial v} \right)_1 - \beta_1 \left(\frac{\partial \alpha}{\partial v} \right)_2,
 \end{aligned}$$

by the help of (iii) and (iv),

$$= - \frac{\partial \Phi}{\partial v_1} + \frac{\partial \Phi}{\partial v_2} \dots \dots \dots (53),$$

and now writing A' for the expression obtained, when the sign of u , v , is changed in A , I shall have also

$$\begin{aligned}
 A' &= (\beta_1 - \beta_2) \left\{ \frac{1}{\beta_1} \left(\frac{\partial \beta}{\partial u} \right)_1 + \frac{1}{\beta_2} \left(\frac{\partial \beta}{\partial u} \right)_2 \right\} \\
 &\quad - (\alpha_1 \beta_2 - \alpha_2 \beta_1) \left\{ \frac{1}{\beta_1} \left(\frac{\partial \beta}{\partial v} \right)_1 + \frac{1}{\beta_2} \left(\frac{\partial \beta}{\partial v} \right)_2 \right\} \\
 &= - \frac{\partial \gamma}{\partial v_1} + \frac{\partial \gamma}{\partial v_2} + \alpha_2 \left(\frac{\partial \beta}{\partial v} \right)_1 - \alpha_1 \left(\frac{\partial \beta}{\partial v} \right)_2 - \beta_2 \left(\frac{\partial \alpha}{\partial v} \right)_1 + \beta_1 \left(\frac{\partial \alpha}{\partial v} \right)_2 \\
 &= - \frac{\partial \Phi}{\partial v_1} - \frac{\partial \Phi}{\partial v_2} \dots \dots \dots (54).
 \end{aligned}$$

Further

$$\begin{aligned}
 AA' &= \left\{ (\beta_1 - \beta_2) \frac{1}{\beta_1} \left(\frac{\partial \beta}{\partial u} \right)_1 - (\alpha_1 \beta_2 - \alpha_2 \beta_1) \frac{1}{\beta_1} \left(\frac{\partial \beta}{\partial v} \right)_1 \right\}^2 \\
 &\quad - \left\{ (\beta_1 - \beta_2) \frac{1}{\beta_2} \left(\frac{\partial \beta}{\partial u} \right)_2 - (\alpha_1 \beta_2 - \alpha_2 \beta_1) \frac{1}{\beta_2} \left(\frac{\partial \beta}{\partial v} \right)_2 \right\}^2 \\
 &= (\beta_1 - \beta_2)^2 \left\{ \frac{1}{\beta_1^2} \left(\frac{\partial \beta}{\partial u} \right)_1^2 - \frac{1}{\beta_2^2} \left(\frac{\partial \beta}{\partial u} \right)_2^2 \right\} \\
 &\quad + (\alpha_1 \beta_2 - \alpha_2 \beta_1)^2 \left\{ \frac{1}{\beta_1^2} \left(\frac{\partial \beta}{\partial v} \right)_1^2 - \frac{1}{\beta_2^2} \left(\frac{\partial \beta}{\partial v} \right)_2^2 \right\} \\
 &\quad - (\beta_1 - \beta_2) (\alpha_1 \beta_2 - \alpha_2 \beta_1) \left\{ \frac{2}{\beta_1^2} \left(\frac{\partial \beta}{\partial u} \right)_1 \left(\frac{\partial \beta}{\partial v} \right)_1 - \frac{2}{\beta_2^2} \left(\frac{\partial \beta}{\partial u} \right)_2 \left(\frac{\partial \beta}{\partial v} \right)_2 \right\}.
 \end{aligned}$$

From which, substituting from (vi), (vii), and (viii),

$$\begin{aligned}
 AA' &= (\beta_1 - \beta_2)' \left(\gamma_1 - \gamma_2 + \frac{\alpha_1}{\beta_1} - \frac{\alpha_2}{\beta_2} \right) \\
 &\quad + (\alpha_1 \beta_2 - \alpha_2 \beta_1)' \left(\alpha_1 - \alpha_2 + \frac{\gamma_1}{\beta_1} - \frac{\gamma_2}{\beta_2} \right) \\
 &\quad + (\beta_1 - \beta_2) (\alpha_1 \beta_2 - \alpha_2 \beta_1) \left(\frac{\alpha_1 \gamma_1}{\beta_1} - \frac{\alpha_2 \gamma_2}{\beta_2} + \frac{1}{\beta_1} - \frac{1}{\beta_2} + \beta_1 - \beta_2 \right) \\
 &= (\gamma_1 - \gamma_2 + \alpha_1 \beta_2 - \alpha_2 \beta_1) \{ (\beta_1 - \beta_2)' + (\alpha_1 - \alpha_2) (\alpha_1 \beta_2 - \alpha_2 \beta_1) \} \\
 &= E\Phi \dots\dots\dots (55).
 \end{aligned}$$

§ 15. Similarly

$$\begin{aligned}
 -D &= (\alpha_1 - \alpha_2) \left\{ \beta_2 \left(\frac{\partial \beta}{\partial u} \right)_1 - \beta_1 \left(\frac{\partial \beta}{\partial u} \right)_2 \right\} \\
 &\quad + (\beta_1 - \beta_2) \left\{ \beta_2 \left(\frac{\partial \beta}{\partial v} \right)_1 - \beta_1 \left(\frac{\partial \beta}{\partial v} \right)_2 \right\}, \\
 -D' &= (\alpha_1 - \alpha_2) \left\{ \beta_2 \left(\frac{\partial \beta}{\partial u} \right)_1 + \beta_1 \left(\frac{\partial \beta}{\partial u} \right)_2 \right\} \\
 &\quad + (\beta_1 - \beta_2) \left\{ \beta_2 \left(\frac{\partial \beta}{\partial v} \right)_1 + \beta_1 \left(\frac{\partial \beta}{\partial v} \right)_2 \right\}.
 \end{aligned}$$

But

$$\begin{aligned}
 \frac{\partial \Phi'}{\partial u_1} &= \left(\frac{\partial \alpha}{\partial u} \right)_1 - \gamma_2 \left(\frac{\partial \beta}{\partial u} \right)_1 + \beta_2 \left(\frac{\partial \gamma}{\partial u} \right)_1 \\
 &= - \left(\frac{\partial \beta}{\partial v} \right)_1 - \gamma_2 \left(\frac{\partial \beta}{\partial u} \right)_1 + \beta_2 \left\{ \frac{1}{\beta_1} \left(\frac{\partial \beta}{\partial v} \right)_1 + \frac{\gamma_1}{\beta_1} \left(\frac{\partial \beta}{\partial u} \right)_1 \right\} \\
 &= - (\beta_1 - \beta_2) \frac{1}{\beta_1} \left(\frac{\partial \beta}{\partial v} \right)_1 + (\gamma_1 \beta_2 - \gamma_2 \beta_1) \frac{1}{\beta_1} \left(\frac{\partial \beta}{\partial u} \right)_1,
 \end{aligned}$$

and also

$$\frac{\partial \Phi'}{\partial u_2} = - (\beta_1 - \beta_2) \frac{1}{\beta_2} \left(\frac{\partial \beta}{\partial v} \right)_2 + (\gamma_1 \beta_2 - \gamma_2 \beta_1) \frac{1}{\beta_2} \left(\frac{\partial \beta}{\partial u} \right)_2,$$

therefore

$$-D = \beta_1 \beta_2 \left(- \frac{\partial \Phi'}{\partial u_1} + \frac{\partial \Phi'}{\partial u_2} \right) + \beta_1 \beta_2 \Phi' \left\{ \frac{1}{\beta_1} \left(\frac{\partial \beta}{\partial u} \right)_1 - \frac{1}{\beta_2} \left(\frac{\partial \beta}{\partial u} \right)_2 \right\},$$

$$\text{or} \quad D = \beta_1' \beta_2' \left\{ \frac{\partial}{\partial u_1} \left(\frac{\Phi'}{\beta_1 \beta_2} \right) - \frac{\partial}{\partial u_2} \left(\frac{\Phi'}{\beta_1 \beta_2} \right) \right\} \dots\dots (56),$$

and

$$-D' = \beta_1 \beta_2 \left(-\frac{\partial \Phi'}{\partial u_1} - \frac{\partial \Phi'}{\partial u_2} \right) + \beta_1 \beta_2 \Phi' \left\{ \frac{1}{\beta_1} \left(\frac{\partial \beta}{\partial u} \right)_1 + \frac{1}{\beta_2} \left(\frac{\partial \beta}{\partial u} \right)_2 \right\},$$

$$\text{or} \quad D' = \beta_1 \beta_2 \left\{ \frac{\partial}{\partial u_1} \left(\frac{\Phi'}{\beta_1 \beta_2} \right) + \frac{\partial}{\partial u_2} \left(\frac{\Phi'}{\beta_1 \beta_2} \right) \right\} \dots \dots (57).$$

Further, by the help of (vi), (vii), and (viii),

$$\begin{aligned} \frac{DD'}{\beta_1 \beta_2} &= (\alpha_1 - \alpha_2)^2 \left(\gamma_1 - \gamma_2 + \frac{\alpha_1}{\beta_1} - \frac{\alpha_2}{\beta_2} \right) \\ &\quad + (\beta_1 - \beta_2)^2 \left(\alpha_1 - \alpha_2 + \frac{\gamma_1}{\beta_1} - \frac{\gamma_2}{\beta_2} \right) \\ &\quad - (\alpha_1 - \alpha_2) (\beta_1 - \beta_2) \left(\frac{\alpha_1 \gamma_1}{\beta_1} - \frac{\alpha_2 \gamma_2}{\beta_2} + \frac{1}{\beta_1} - \frac{1}{\beta_2} + \beta_1 - \beta_2 \right) \end{aligned}$$

$$= \frac{1}{\beta_1 \beta_2} (\alpha_1 - \alpha_2 + \gamma_1 \beta_2 - \gamma_2 \beta_1) \{ (\beta_1 - \beta_2)^2 + (\alpha_1 - \alpha_2) (\alpha_1 \beta_2 - \alpha_2 \beta_1) \},$$

$$\text{or} \quad DD' = \beta_1 \beta_2 E \Phi' \dots \dots \dots (58).$$

Also since

$$\begin{aligned} -\frac{D}{\beta_1 \beta_2} &= (\alpha_1 - \alpha_2) \left\{ \frac{1}{\beta_1} \left(\frac{\partial \beta}{\partial u} \right)_1 - \frac{1}{\beta_2} \left(\frac{\partial \beta}{\partial u} \right)_2 \right\} \\ &\quad + (\beta_1 - \beta_2) \left\{ \frac{1}{\beta_1} \left(\frac{\partial \beta}{\partial v} \right)_1 - \frac{1}{\beta_2} \left(\frac{\partial \beta}{\partial v} \right)_2 \right\}, \end{aligned}$$

$$\begin{aligned} A' &= (\beta_1 - \beta_2) \left\{ \frac{1}{\beta_1} \left(\frac{\partial \beta}{\partial u} \right)_1 + \frac{1}{\beta_2} \left(\frac{\partial \beta}{\partial u} \right)_2 \right\} \\ &\quad - (\alpha_1 \beta_2 - \alpha_2 \beta_1) \left\{ \frac{1}{\beta_1} \left(\frac{\partial \beta}{\partial v} \right)_1 + \frac{1}{\beta_2} \left(\frac{\partial \beta}{\partial v} \right)_2 \right\}, \end{aligned}$$

$$\begin{aligned} -\frac{DA'}{\beta_1 \beta_2} &= (\alpha_1 - \alpha_2) (\beta_1 - \beta_2) \left\{ \frac{1}{\beta_1^2} \left(\frac{\partial \beta}{\partial u} \right)_1^2 - \frac{1}{\beta_2^2} \left(\frac{\partial \beta}{\partial u} \right)_2^2 \right\} \\ &\quad - (\alpha_1 \beta_2 - \alpha_2 \beta_1) (\beta_1 - \beta_2) \left\{ \frac{1}{\beta_1^2} \left(\frac{\partial \beta}{\partial v} \right)_1^2 - \frac{1}{\beta_2^2} \left(\frac{\partial \beta}{\partial v} \right)_2^2 \right\} \end{aligned}$$

$$\begin{aligned} &- (\alpha_1 - \alpha_2) (\alpha_1 \beta_2 - \alpha_2 \beta_1) \left\{ \frac{1}{\beta_1^2} \left(\frac{\partial \beta}{\partial u} \right)_1 \left(\frac{\partial \beta}{\partial v} \right)_1 - \frac{1}{\beta_2^2} \left(\frac{\partial \beta}{\partial u} \right)_2 \left(\frac{\partial \beta}{\partial v} \right)_2 \right. \\ &\quad \left. + \frac{1}{\beta_1 \beta_2} \left(\left(\frac{\partial \beta}{\partial u} \right)_1 \left(\frac{\partial \beta}{\partial v} \right)_2 - \left(\frac{\partial \beta}{\partial u} \right)_2 \left(\frac{\partial \beta}{\partial v} \right)_1 \right) \right\} \end{aligned}$$

$$+ (\beta_1 - \beta_2)^2 \left\{ \frac{1}{\beta_1^2} \left(\frac{\partial \beta}{\partial u} \right)_1 \left(\frac{\partial \beta}{\partial v} \right)_1 - \frac{1}{\beta_2^2} \left(\frac{\partial \beta}{\partial u} \right)_2 \left(\frac{\partial \beta}{\partial v} \right)_2 \right. \\ \left. - \frac{1}{\beta_1 \beta_2} \left(\left(\frac{\partial \beta}{\partial u} \right)_1 \left(\frac{\partial \beta}{\partial v} \right)_2 - \left(\frac{\partial \beta}{\partial u} \right)_2 \left(\frac{\partial \beta}{\partial v} \right)_1 \right) \right\},$$

Therefore

$$\frac{1}{2} (DA' - D'A) = \{(\beta_1 - \beta_2)^2 + (\alpha_1 - \alpha_2) (\alpha_1 \beta_2 - \alpha_2 \beta_1)\} \\ \times \left\{ \left(\frac{\partial \beta}{\partial u} \right)_1 \left(\frac{\partial \beta}{\partial v} \right)_2 - \left(\frac{\partial \beta}{\partial u} \right)_2 \left(\frac{\partial \beta}{\partial v} \right)_1 \right\},$$

$$\text{or } DA' - D'A = 2E \left\{ \left(\frac{\partial \beta}{\partial u} \right)_1 \left(\frac{\partial \beta}{\partial v} \right)_2 - \left(\frac{\partial \beta}{\partial u} \right)_2 \left(\frac{\partial \beta}{\partial v} \right)_1 \right\} \dots (59),$$

And using (vi), (vii), and (viii),

$$- \frac{DA' + D'A}{2\beta_1 \beta_2} = (\alpha_1 - \alpha_2) (\beta_1 - \beta_2) \left(\gamma_1 - \gamma_2 + \frac{\alpha_1}{\beta_1} - \frac{\alpha_2}{\beta_2} \right) \\ - (\alpha_1 \beta_2 - \alpha_2 \beta_1) (\beta_1 - \beta_2) \left(\alpha_1 - \alpha_2 + \frac{\gamma_1}{\beta_1} - \frac{\gamma_2}{\beta_2} \right) \\ + \frac{1}{2} \{ (\alpha_1 - \alpha_2) (\alpha_1 \beta_2 - \alpha_2 \beta_1) - (\beta_1 - \beta_2)^2 \} \\ \times \left\{ \frac{\alpha_1 \gamma_1}{\beta_1^2} - \frac{\alpha_2 \gamma_2}{\beta_2^2} + \frac{1}{\beta_1} - \frac{1}{\beta_2} + \beta_1 - \beta_2 \right\} \\ = \frac{1}{2\beta_1 \beta_2} \{ (\beta_1 - \beta_2)^2 + (\alpha_1 - \alpha_2) (\alpha_1 \beta_2 - \alpha_2 \beta_1) \} \\ \times \{ \alpha_1 \gamma_1 \beta_2 - \alpha_2 \gamma_2 \beta_1 + (\beta_1 - \beta_2) (1 - \beta_1 \beta_2) \}, \\ \text{or } DA' + D'A = -E \{ \alpha_1 \gamma_1 \beta_2 - \alpha_2 \gamma_2 \beta_1 + (\beta_1 - \beta_2) (1 - \beta_1 \beta_2) \} \\ = -E\Delta \dots (60),$$

But I have also

$$\beta_1 \beta_2 \left[\frac{\partial \Phi}{\partial v_1} \cdot \frac{\partial \Phi}{\partial u_2} - \Phi \frac{\partial^2 \Phi}{\partial v_1 \partial u_2} \right] \\ = -\beta_1 \beta_2 \left\{ \left(\frac{\partial \gamma}{\partial v} \right)_1 + \beta_2 \left(\frac{\partial \alpha}{\partial v} \right)_1 - \alpha_2 \left(\frac{\partial \beta}{\partial v} \right)_1 \right\} \\ \times \left\{ \left(\frac{\partial \gamma}{\partial u} \right)_2 + \beta_1 \left(\frac{\partial \alpha}{\partial u} \right)_2 - \alpha_1 \left(\frac{\partial \beta}{\partial u} \right)_2 \right\} \\ - \beta_1 \beta_2 (\gamma_1 - \gamma_2 + \alpha_1 \beta_2 - \alpha_2 \beta_1) \left\{ \left(\frac{\partial \alpha}{\partial v} \right)_1 \left(\frac{\partial \beta}{\partial u} \right)_2 - \left(\frac{\partial \alpha}{\partial u} \right)_2 \left(\frac{\partial \beta}{\partial v} \right)_1 \right\}$$

$$\begin{aligned}
 &= - \left\{ - (\beta_1 - \beta_2) \left(\frac{\partial \beta}{\partial u} \right)_1 + (\alpha_1 \beta_2 - \alpha_2 \beta_1) \left(\frac{\partial \beta}{\partial v} \right)_1 \right\} \\
 &\quad \times \left\{ (\gamma_1 - \alpha_1 \beta_2) \left(\frac{\partial \beta}{\partial u} \right)_2 + (1 - \beta_1 \beta_2) \left(\frac{\partial \beta}{\partial v} \right)_2 \right\} \\
 &\quad - (\gamma_1 - \gamma_2 + \alpha_1 \beta_2 - \alpha_2 \beta_1) \left[\beta_2 \left(\frac{\partial \beta}{\partial u} \right)_2 \left\{ \alpha_1 \left(\frac{\partial \beta}{\partial v} \right)_1 + \left(\frac{\partial \beta}{\partial u} \right)_1 \right\} \right. \\
 &\quad \left. + \beta_1 \beta_2 \left(\frac{\partial \beta}{\partial v} \right)_1 \left(\frac{\partial \beta}{\partial v} \right)_2 \right] \\
 &= \left(\frac{\partial \beta}{\partial u} \right)_1 \left(\frac{\partial \beta}{\partial u} \right)_2 \{ \beta_1 \gamma_2 - \beta_2 \gamma_1 - \beta_1 \beta_2 (\alpha_1 - \alpha_2) \} \\
 &\quad + \left(\frac{\partial \beta}{\partial u} \right)_1 \left(\frac{\partial \beta}{\partial v} \right)_2 (\beta_1 - \beta_2) (1 - \beta_1 \beta_2) \\
 &\quad - \left(\frac{\partial \beta}{\partial v} \right)_1 \left(\frac{\partial \beta}{\partial u} \right)_2 (\alpha_1 \gamma_2 \beta_2 - \alpha_2 \gamma_1 \beta_1) \\
 &\quad - \left(\frac{\partial \beta}{\partial v} \right)_1 \left(\frac{\partial \beta}{\partial v} \right)_2 \{ \alpha_1 \beta_2 - \alpha_2 \beta_1 + \beta_1 \beta_2 (\gamma_1 - \gamma_2) \}.
 \end{aligned}$$

The first term of this may be written

$$\beta_1 \beta_2 \left(\frac{\partial \beta}{\partial u} \right)_1 \left(\frac{\partial \beta}{\partial u} \right)_2 \left\{ \alpha_2 + \frac{\gamma_2}{\beta_2} + b - \left(\alpha_1 + \frac{\gamma_1}{\beta_1} + b \right) \right\};$$

which, by the help of (xi), is transformed into

$$\begin{aligned}
 &\frac{1}{2} \left(\frac{\partial \beta}{\partial u} \right)_1 \left(\frac{\partial \beta}{\partial v} \right)_2 \beta_1 \beta_2 \left(c - \frac{\alpha_2 \gamma_2}{\beta_2} - \frac{1}{\beta_2} - \beta_2 \right), \\
 &- \frac{1}{2} \left(\frac{\partial \beta}{\partial v} \right)_1 \left(\frac{\partial \beta}{\partial u} \right)_2 \beta_1 \beta_2 \left(c - \frac{\alpha_1 \gamma_1}{\beta_1} - \frac{1}{\beta_1} - \beta_1 \right).
 \end{aligned}$$

Similarly the fourth term is transformed into

$$\begin{aligned}
 &- \frac{1}{2} \left(\frac{\partial \beta}{\partial u} \right)_1 \left(\frac{\partial \beta}{\partial v} \right)_2 \beta_1 \beta_2 \left(c - \frac{\alpha_1 \gamma_1}{\beta_1} - \frac{1}{\beta_1} - \beta_1 \right), \\
 &+ \frac{1}{2} \left(\frac{\partial \beta}{\partial v} \right)_1 \left(\frac{\partial \beta}{\partial u} \right)_2 \beta_1 \beta_2 \left(c - \frac{\alpha_2 \gamma_2}{\beta_2} - \frac{1}{\beta_2} - \beta_2 \right).
 \end{aligned}$$

Using these transformations I get

$$\begin{aligned}
 \beta_1 \beta_2 \left[\frac{\partial \Phi}{\partial v_1} \cdot \frac{\partial \Phi}{\partial u_2} - \Phi \frac{\partial^2 \Phi}{\partial v_1 \partial u_2} \right] &= \frac{1}{2} \left\{ \left(\frac{\partial \beta}{\partial u} \right)_1 \left(\frac{\partial \beta}{\partial v} \right)_2 - \left(\frac{\partial \beta}{\partial v} \right)_1 \left(\frac{\partial \beta}{\partial u} \right)_2 \right\} \\
 &\quad \times \{ \alpha_1 \gamma_2 \beta_2 - \alpha_2 \gamma_1 \beta_1 + (\beta_1 - \beta_2) (1 - \beta_1 \beta_2) \}.
 \end{aligned}$$

From which

$$\begin{aligned}\frac{\partial \Phi}{\partial v_1} \frac{\partial \Phi}{\partial u_2} - \Phi \frac{\partial^2 \Phi}{\partial v_1 \partial u_2} &= - \frac{(DA' - D'A)(DA' + D'A)}{4\beta_1\beta_2 E^2} \\ &= - \frac{D^2 A'^2 - D'^2 A^2}{4\beta_1\beta_2 E^2} \dots\dots\dots (61).\end{aligned}$$

§ 16. I have also, since

$$\begin{aligned}-2B &= A(\alpha_1 + \alpha_2) - D\left(\frac{1}{\beta_1} + \frac{1}{\beta_2}\right) + E\left\{\frac{1}{\beta_1}\left(\frac{\partial \beta}{\partial v}\right)_1 + \frac{1}{\beta_2}\left(\frac{\partial \beta}{\partial v}\right)_2\right\}, \\ -2BA' &= AA'(\alpha_1 + \alpha_2) - DA'\left(\frac{1}{\beta_1} + \frac{1}{\beta_2}\right) \\ &\quad + EA'\left\{\frac{1}{\beta_1}\left(\frac{\partial \beta}{\partial v}\right)_1 + \frac{1}{\beta_2}\left(\frac{\partial \beta}{\partial v}\right)_2\right\} \\ &= E\Phi(\alpha_1 + \alpha_2) \\ &\quad - \left(\frac{1}{\beta_1} + \frac{1}{\beta_2}\right)\left\{-\frac{1}{2}\Delta E + E\left\{\left(\frac{\partial \beta}{\partial u}\right)_1\left(\frac{\partial \beta}{\partial v}\right)_2 - \left(\frac{\partial \beta}{\partial u}\right)_2\left(\frac{\partial \beta}{\partial v}\right)_1\right\}\right\} \\ &\quad + EA'\left\{\frac{1}{\beta_1}\left(\frac{\partial \beta}{\partial v}\right)_1 + \frac{1}{\beta_2}\left(\frac{\partial \beta}{\partial v}\right)_2\right\}, \\ \text{or } \frac{2B}{A} + \alpha_1 + \alpha_2 &= -\frac{1}{\Phi}\left[-\left(\frac{\beta_1 + \beta_2}{\beta_1\beta_2}\right)\left\{\left(\frac{\partial \beta}{\partial u}\right)_1\left(\frac{\partial \beta}{\partial v}\right)_2 - \left(\frac{\partial \beta}{\partial u}\right)_2\left(\frac{\partial \beta}{\partial v}\right)_1\right\}\right. \\ &\quad \left.+ A'\left\{\frac{1}{\beta_1}\left(\frac{\partial \beta}{\partial v}\right)_1 + \frac{1}{\beta_2}\left(\frac{\partial \beta}{\partial v}\right)_2\right\} + \frac{\beta_1 + \beta_2}{2\beta_1\beta_2}\Delta\right] \dots (62).\end{aligned}$$

From this it follows at once that

$$\begin{aligned}-2\frac{BA' - B'A}{E} &= -2\left(\frac{\beta_1 + \beta_2}{\beta_1\beta_2}\right)\left\{\left(\frac{\partial \beta}{\partial u}\right)_1\left(\frac{\partial \beta}{\partial v}\right)_2 - \left(\frac{\partial \beta}{\partial u}\right)_2\left(\frac{\partial \beta}{\partial v}\right)_1\right\} \\ &\quad + 2\left(\frac{\beta_1 - \beta_2}{\beta_1\beta_2}\right)\left\{\left(\frac{\partial \beta}{\partial u}\right)_1\left(\frac{\partial \beta}{\partial v}\right)_2 + \left(\frac{\partial \beta}{\partial u}\right)_2\left(\frac{\partial \beta}{\partial v}\right)_1\right\} \\ &\quad - 4\left(\frac{\alpha_1\beta_2 - \alpha_2\beta_1}{\beta_1\beta_2}\right)\left\{\left(\frac{\partial \beta}{\partial v}\right)_1\left(\frac{\partial \beta}{\partial v}\right)_2\right\} \\ &= -4\left\{\left(\frac{\partial \beta}{\partial v}\right)_1\left(\frac{\partial \alpha}{\partial v}\right)_2 - \left(\frac{\partial \beta}{\partial v}\right)_2\left(\frac{\partial \alpha}{\partial v}\right)_1\right\},\end{aligned}$$

which is $BA' - B'A = 2E \frac{\partial^2 \Phi}{\partial v_1 \partial v_2},$

or $\frac{B}{A} - \frac{B'}{A'} = \frac{2}{\Phi} \frac{\partial^2 \Phi}{\partial v_1 \partial v_2} \dots\dots\dots (63).$

§ 17. Similarly

$$\begin{aligned} -2C &= D \left(\frac{\alpha_1}{\beta_1} + \frac{\alpha_2}{\beta_2} \right) - A(\beta_1 + \beta_2) + E \left\{ \frac{1}{\beta_1} \left(\frac{\partial \beta}{\partial u} \right)_1 + \frac{1}{\beta_2} \left(\frac{\partial \beta}{\partial u} \right)_2 \right\}, \\ \frac{2C}{D} + \frac{\alpha_1}{\beta_1} + \frac{\alpha_2}{\beta_2} &= \frac{1}{DD'} \left[(\beta_1 + \beta_2) AD' \right. \\ &\quad \left. - ED' \left\{ \frac{1}{\beta_1} \left(\frac{\partial \beta}{\partial u} \right)_1 + \frac{1}{\beta_2} \left(\frac{\partial \beta}{\partial u} \right)_2 \right\} \right] \\ &= \frac{1}{\beta_1 \beta_2 \Phi'} \left[-(\beta_1 + \beta_2) \left\{ \frac{1}{2} \Delta + \left(\frac{\partial \beta}{\partial u} \right)_1 \left(\frac{\partial \beta}{\partial v} \right)_2 - \left(\frac{\partial \beta}{\partial u} \right)_2 \left(\frac{\partial \beta}{\partial v} \right)_1 \right\} \right. \\ &\quad \left. - D' \left\{ \frac{1}{\beta_1} \left(\frac{\partial \beta}{\partial u} \right)_1 + \frac{1}{\beta_2} \left(\frac{\partial \beta}{\partial u} \right)_2 \right\} \right] \dots\dots\dots (64), \end{aligned}$$

and therefore

$$\begin{aligned} 2 \frac{CD' - C'D}{E} &= -2(\beta_1 + \beta_2) \left\{ \left(\frac{\partial \beta}{\partial u} \right)_1 \left(\frac{\partial \beta}{\partial v} \right)_2 - \left(\frac{\partial \beta}{\partial u} \right)_2 \left(\frac{\partial \beta}{\partial v} \right)_1 \right\} \\ &\quad + 4(\alpha_1 - \alpha_2) \left(\frac{\partial \beta}{\partial u} \right)_1 \left(\frac{\partial \beta}{\partial u} \right)_2 \\ &\quad + 2(\beta_1 - \beta_2) \left\{ \left(\frac{\partial \beta}{\partial u} \right)_1 \left(\frac{\partial \beta}{\partial v} \right)_2 + \left(\frac{\partial \beta}{\partial u} \right)_2 \left(\frac{\partial \beta}{\partial v} \right)_1 \right\} \\ &= 4\beta_1 \left(\frac{\partial \beta}{\partial u} \right)_2 \left(\frac{\partial \beta}{\partial v} \right)_1 - 4\beta_2 \left(\frac{\partial \beta}{\partial u} \right)_1 \left(\frac{\partial \beta}{\partial v} \right)_2 + 4(\alpha_1 - \alpha_2) \left(\frac{\partial \beta}{\partial u} \right)_1 \left(\frac{\partial \beta}{\partial u} \right)_2. \end{aligned}$$

But $\beta_1 \beta_2 \frac{\partial^2 \Phi'}{\partial u_1 \partial u_2} \left(\frac{\Phi'}{\beta_1 \beta_2} \right)$

$$= \beta_1 \beta_2 \frac{\partial^2 \Phi'}{\partial u_1 \partial u_2} - \beta_1 \left(\frac{\partial \beta}{\partial u} \right)_2 \frac{\partial \Phi'}{\partial u_1} - \beta_2 \left(\frac{\partial \beta}{\partial u} \right)_1 \frac{\partial \Phi'}{\partial u_2} + \Phi' \left(\frac{\partial \beta}{\partial u} \right)_1 \left(\frac{\partial \beta}{\partial u} \right)_2,$$

and $\frac{\partial^2 \Phi'}{\partial u_1 \partial u_2} = \left(\frac{\partial \gamma}{\partial u} \right)_1 \left(\frac{\partial \beta}{\partial u} \right)_2 - \left(\frac{\partial \gamma}{\partial u} \right)_2 \left(\frac{\partial \beta}{\partial u} \right)_1,$

or $\beta_1 \beta_2 \frac{\partial^2 \Phi'}{\partial u_1 \partial u_2} = \beta_2 \left(\frac{\partial \beta}{\partial u} \right)_2 \left(\frac{\partial \beta}{\partial v} \right)_1 - \beta_1 \left(\frac{\partial \beta}{\partial u} \right)_1 \left(\frac{\partial \beta}{\partial v} \right)_2$
 $+ (\beta_2 \gamma_1 - \beta_1 \gamma_2) \left(\frac{\partial \beta}{\partial u} \right)_1 \left(\frac{\partial \beta}{\partial u} \right)_2,$

by (v), therefore

$$\beta_1 \beta_2 \frac{\partial^2}{\partial u_1 \partial u_2} \left(\frac{\Phi'}{\beta_1 \beta_2} \right) = \beta_1 \left(\frac{\partial \beta}{\partial u} \right)_2 \left(\frac{\partial \beta}{\partial v} \right)_1 - \beta_2 \left(\frac{\partial \beta}{\partial u} \right)_1 \left(\frac{\partial \beta}{\partial v} \right)_2 \\ + (\alpha_1 - \alpha_2) \left(\frac{\partial \beta}{\partial u} \right)_1 \left(\frac{\partial \beta}{\partial u} \right)_2.$$

From which I get

$$CD' - C'D = 2E\beta_1 \beta_2 \frac{\partial^2}{\partial u_1 \partial u_2} \left(\frac{\Phi'}{\beta_1 \beta_2} \right) \left\{ \begin{array}{l} \text{or} \\ \frac{C}{D} - \frac{C'}{D'} = \frac{2\beta_1 \beta_2}{\Phi'} \frac{\partial^2}{\partial u_1 \partial u_2} \left(\frac{\Phi'}{\beta_1 \beta_2} \right) \end{array} \right\} \dots (65).$$

I have also, using (62),

$$\frac{B}{A} + \frac{B'}{A'} + \alpha_1 + \alpha_2 = \frac{1}{\beta_1 \Phi} \left(\frac{\partial \beta}{\partial v} \right)_1 \frac{\partial \Phi}{\partial v_1} + \frac{1}{\beta_2 \Phi} \left(\frac{\partial \beta}{\partial v} \right)_2 \frac{\partial \Phi}{\partial v_2} - \frac{\beta_1 + \beta_2}{2\beta_1 \beta_2} \frac{\Delta}{\Phi} \\ \dots\dots\dots (66).$$

and using (64)

$$\frac{C}{D} + \frac{C'}{D'} + \frac{\alpha_1}{\beta_1} + \frac{\alpha_2}{\beta_2} = -\frac{\beta_1 \beta_2}{\Phi'} \left\{ \frac{1}{\beta_1} \left(\frac{\partial \beta}{\partial u} \right)_1 \frac{\partial}{\partial u_1} \left(\frac{\Phi'}{\beta_1 \beta_2} \right) \right. \\ \left. + \frac{1}{\beta_2} \left(\frac{\partial \beta}{\partial u} \right)_2 \frac{\partial}{\partial u_2} \left(\frac{\Phi'}{\beta_1 \beta_2} \right) \right\} - \frac{\beta_1 + \beta_2}{2\beta_1 \beta_2} \frac{\Delta}{\Phi'} \dots\dots\dots (67).$$

§ 18. Returning now to the addition theorem, I write

$$u, v = u_1 + u_2, v_1 + v_2; \quad u', v' = u_1 - u_2, v_1 - v_2;$$

and $\beta = \beta(u, v), \beta' = \beta(u', v'), \&c., \&c.,$

and from § 13 (49) I get at once

$$\beta \beta_1 \beta_2 = \frac{D^2}{A^2} = \frac{(DA')^2}{(AA')^2} \\ = \frac{1}{4\Phi^2} \left[\Delta - 2 \left\{ \left(\frac{\partial \beta}{\partial u} \right)_1 \left(\frac{\partial \beta}{\partial v} \right)_2 - \left(\frac{\partial \beta}{\partial u} \right)_2 \left(\frac{\partial \beta}{\partial v} \right)_1 \right\} \right]^2 \\ = \frac{1}{4\Delta^2 \Phi^2} \left[\Delta^2 - 4\beta_1 \beta_2 \left\{ \frac{\partial \Phi}{\partial v_1} \frac{\partial \Phi}{\partial u_2} - \Phi \frac{\partial^2 \Phi}{\partial v_1 \partial u_2} \right\} \right]^2.$$

$$\text{Also} \quad \beta \beta' \beta_1 \beta_2 = \frac{(DD')^2}{(AA')^2} = \frac{(\beta_1 \beta_2 E \Phi')^2}{(E \Phi)^2} \dots\dots\dots (68).$$

$$\text{Therefore} \quad \beta \beta' = \frac{\Phi'^2}{\Phi^2} \dots\dots\dots (69).$$

$$\text{Further } \beta_1 \beta_2 (\beta - \beta') = \frac{D^2 A'^2 - D'^2 A^2}{A^2 A'^2},$$

that is

$$\begin{aligned} \beta - \beta' &= -\frac{4}{\Phi^2} \left\{ \frac{\partial \Phi}{\partial v_1} \frac{\partial \Phi}{\partial u_2} - \Phi \frac{\partial^2 \Phi}{\partial v_1 \partial u_2} \right\} \\ &= -\frac{4}{\Phi^2} \left\{ \frac{\partial \Phi}{\partial v_2} \frac{\partial \Phi}{\partial u_1} - \Phi \frac{\partial^2 \Phi}{\partial v_2 \partial u_1} \right\} \\ &= 4 \frac{\partial^2}{\partial v_1 \partial u_2} (\log \Phi) = 4 \frac{\partial^2}{\partial v_2 \partial u_1} (\log \Phi) \dots (70), \end{aligned}$$

since $\beta - \beta'$ is unaltered by the interchange of the suffixes 1 and 2.

I get also from (47), (63) and (66)

$$\begin{aligned} \alpha &= \frac{E^2}{A^2} - 2 \frac{B}{A} - \alpha_1 - \alpha_2 \\ &= \frac{1}{\Phi^2} \left(\frac{\partial \Phi}{\partial v_1} + \frac{\partial \Phi}{\partial v_2} \right)^2 - \frac{2}{\Phi} \frac{\partial^2 \Phi}{\partial v_1 \partial v_2} + \frac{\beta_1 + \beta_2}{2\beta_1 \beta_2} \frac{\Delta}{\Phi} \\ &\quad - \frac{1}{\beta_1} \left(\frac{\partial \beta}{\partial v} \right)_1 \frac{1}{\Phi} \frac{\partial \Phi}{\partial v_1} - \frac{1}{\beta_2} \left(\frac{\partial \beta}{\partial v} \right)_2 \frac{1}{\Phi} \frac{\partial \Phi}{\partial v_2}, \end{aligned}$$

from which it follows that

$$\begin{aligned} \alpha - \alpha' &= \frac{4}{\Phi^2} \left\{ \frac{\partial \Phi}{\partial v_1} \cdot \frac{\partial \Phi}{\partial v_2} - \Phi \frac{\partial^2 \Phi}{\partial v_1 \partial v_2} \right\} \\ &= -4 \frac{\partial^2}{\partial v_1 \partial v_2} (\log \Phi) \dots \dots \dots (71). \end{aligned}$$

Similarly, I shall get from (48), (65) and (67)

$$\begin{aligned} \frac{\alpha}{\beta} &= \frac{E^2}{D^2} - \frac{2C}{D} - \frac{\alpha_1}{\beta_1} - \frac{\alpha_2}{\beta_2} \\ &= \frac{\beta_1^2 \beta_2^2}{\Phi'^2} \left\{ \frac{\partial}{\partial u_1} \left(\frac{\Phi'}{\beta_1 \beta_2} \right) + \frac{\partial}{\partial u_2} \left(\frac{\Phi'}{\beta_1 \beta_2} \right) \right\}^2 - \frac{2\beta_1 \beta_2}{\Phi'} \frac{\partial^2}{\partial u_1 \partial u_2} \left(\frac{\Phi'}{\beta_1 \beta_2} \right) \\ &\quad + \frac{\beta_1 + \beta_2}{2\beta_1 \beta_2} \frac{\Delta}{\Phi'} + \frac{\beta_1 \beta_2}{\Phi'} \left\{ \frac{1}{\beta_1} \left(\frac{\partial \beta}{\partial u} \right)_1 \frac{\partial}{\partial u_1} \left(\frac{\Phi'}{\beta_1 \beta_2} \right) + \frac{1}{\beta_2} \left(\frac{\partial \beta}{\partial u} \right)_2 \frac{\partial}{\partial u_2} \left(\frac{\Phi'}{\beta_1 \beta_2} \right) \right\} \\ &= \frac{1}{\Phi'^2} \left(\frac{\partial \Phi'}{\partial u_1} + \frac{\partial \Phi'}{\partial u_2} \right)^2 - \frac{2}{\Phi'} \frac{\partial^2 \Phi'}{\partial u_1 \partial u_2} + \frac{\beta_1 + \beta_2}{2\beta_1 \beta_2} \frac{\Delta}{\Phi'} \\ &\quad - \frac{1}{\beta_1} \left(\frac{\partial \beta}{\partial u} \right)_1 \frac{1}{\Phi'} \frac{\partial \Phi'}{\partial u_1} - \frac{1}{\beta_2} \left(\frac{\partial \beta}{\partial u} \right)_2 \frac{1}{\Phi'} \frac{\partial \Phi'}{\partial u_2}, * \end{aligned}$$

* This reduction is most easily obtained by adding the half period ω to both sets of arguments (u_1, v_1) and (u_2, v_2) .

and

$$\frac{\alpha}{\beta} - \frac{\alpha'}{\beta'} = \frac{4\beta_1^2\beta_2^2}{\Phi'^2} \left\{ \frac{\partial}{\partial u_1} \left(\frac{\Phi'}{\beta_1\beta_2} \right) \frac{\partial}{\partial u_2} \left(\frac{\Phi'}{\beta_1\beta_2} \right) - \frac{\Phi'}{\beta_1\beta_2} \frac{\partial^2}{\partial u_1 \partial u_2} \left(\frac{\Phi'}{\beta_1\beta_2} \right) \right\} \\ = -4 \frac{\partial^2}{\partial u_1 \partial u_2} \log \left(\frac{\Phi'}{\beta_1\beta_2} \right) = -4 \frac{\partial^2}{\partial u_1 \partial u_2} (\log \Phi') \dots\dots(72).$$

§ 19. Let us now consider the formulæ to be deduced from these by adding the half period ω to one set of arguments, say to (u_1, v_1) , so that α_1 becomes γ_1/β_1 , β_1 becomes $1/\beta_1$, and γ_1 becomes α_1/β_1 with corresponding changes for α, β, γ and α', β', γ'

Making this change, Φ becomes Φ'/β_1 ,

$$\Phi' \quad \text{,,} \quad \Phi/\beta_1,$$

$$\Delta \quad \text{,,} \quad \Delta/\beta_1^2,$$

and I deduce

$$\frac{\beta_1\beta_2}{\beta} = \frac{\left[\Delta + 2 \left\{ \left(\frac{\partial\beta}{\partial u} \right)_1 \left(\frac{\partial\beta}{\partial v} \right)_2 - \left(\frac{\partial\beta}{\partial u} \right)_2 \left(\frac{\partial\beta}{\partial v} \right)_1 \right\} \right]^2}{4\Phi'^2} \dots(73),$$

$$\frac{1}{\beta} - \frac{1}{\beta'} = 4 \frac{\partial^2}{\partial v_1 \partial u_2} (\log \Phi') = 4 \frac{\partial^2}{\partial v_2 \partial u_1} (\log \Phi') \dots(74),$$

$$\frac{\gamma}{\beta} = \frac{1}{\Phi'^2} \left(\frac{\partial\Phi'}{\partial v_1} + \frac{\partial\Phi'}{\partial v_2} \right)^2 - \frac{2}{\Phi'} \frac{\partial^2\Phi'}{\partial v_1 \partial v_2} + \frac{1+\beta_1\beta_2}{2\beta_1\beta_2} \frac{\Delta}{\Phi'} \\ - \frac{1}{\beta_1} \left(\frac{\partial\beta}{\partial v} \right)_1 \frac{1}{\Phi'} \frac{\partial\Phi'}{\partial v_1} - \frac{1}{\beta_2} \left(\frac{\partial\beta}{\partial v} \right)_2 \frac{1}{\Phi'} \frac{\partial\Phi'}{\partial v_2},$$

$$\frac{\gamma}{\beta} - \frac{\gamma'}{\beta'} = -4 \frac{\partial^2}{\partial v_1 \partial v_2} (\log \Phi') \dots\dots\dots(75),$$

$$\gamma = \frac{1}{\Phi'^2} \left(\frac{\partial\Phi}{\partial u_1} + \frac{\partial\Phi}{\partial u_2} \right)^2 - \frac{2}{\Phi} \frac{\partial^2\Phi}{\partial u_1 \partial u_2} + \frac{1+\beta_1\beta_2}{2\beta_1\beta_2} \frac{\Delta}{\Phi} \\ - \frac{1}{\beta_1} \left(\frac{\partial\beta}{\partial u} \right)_1 \frac{1}{\Phi} \frac{\partial\Phi}{\partial u_1} - \frac{1}{\beta_2} \left(\frac{\partial\beta}{\partial u} \right)_2 \frac{1}{\Phi} \frac{\partial\Phi}{\partial u_2},$$

$$\gamma - \gamma' = -4 \frac{\partial^2}{\partial u_1 \partial u_2} (\log \Phi) \dots\dots\dots(76).$$

II.

Integrals of the second kind.

§ 20. From the equations

$$\alpha - \alpha' = -4 \frac{\partial^2}{\partial v_1 \partial v_2} (\log \Phi),$$

$$\beta - \beta' = 4 \frac{\partial^2}{\partial v_1 \partial u_2} (\log \Phi)$$

I get

$$\int -(\alpha - \alpha') dv_2 + (\beta - \beta') du_2 = 4 \frac{\partial}{\partial v_1} (\log \Phi),$$

$$\text{or } Z(u, v) + Z(u', v') - 2Z(u_1, v_1) = \frac{\partial}{\partial v_1} (\log \Phi)$$

$$= \frac{1}{\Phi} \frac{\partial \Phi}{\partial v_1} = \frac{\left(\frac{\partial \gamma}{\partial v}\right)_1 + \beta_2 \left(\frac{\partial \alpha}{\partial v}\right)_1 - \alpha_2 \left(\frac{\partial \beta}{\partial v}\right)_1}{\gamma_1 - \gamma_2 + \alpha_1 \beta_2 - \alpha_2 \beta_1} \dots\dots\dots(1),$$

both sides vanishing when $u_2, v_2 = 0, 0$.

Also interchanging u_1, v_1 and u_2, v_2 , I get

$$Z(u, v) - Z(u', v') - 2Z(u_2, v_2) = \frac{1}{\Phi} \frac{\partial \Phi}{\partial v_2} \dots\dots(2).$$

From (1) and (2), by addition,

$$Z(u, v) - Z(u_1, v_1) - Z(u_2, v_2) = \frac{1}{2\Phi} \left(\frac{\partial \Phi}{\partial v_1} + \frac{\partial \Phi}{\partial v_2} \right) \dots\dots(3).$$

Similarly, from the equations,

$$\beta - \beta' = 4 \frac{\partial^2}{\partial u_1 \partial v_2} (\log \Phi),$$

$$\gamma - \gamma' = -4 \frac{\partial^2}{\partial u_1 \partial u_2} (\log \Phi)$$

I get

$$\int (\beta - \beta') dv_2 - (\gamma - \gamma') du_2 = 4 \frac{\partial}{\partial u_1} (\log \Phi),$$

that is

$$\zeta(u, v) + \zeta(u', v') - 2\zeta(u_1, v_1) = \frac{\partial}{\partial u_1} (\log \Phi) \dots\dots(4),$$

and I shall also get

$$\zeta(u, v) - \zeta(u_1, v_1) - \zeta(u_2, v_2) = \frac{1}{2\Phi} \left(\frac{\partial\Phi}{\partial u_1} + \frac{\partial\Phi}{\partial u_2} \right) \dots (5).$$

Integrals of the third kind.

§ 21. From (1) and (4) I get, by integrating with respect to v_2 and u_2 ,

$$\begin{aligned} \int_{0,0} \frac{1}{\Phi} \left(\frac{\partial\Phi}{\partial v_1} dv_2 + \frac{\partial\Phi}{\partial u_1} du_2 \right) &= -2v_2 Z(u_1, v_1) - 2u_2 \zeta(u_1, v_1) \\ &\quad + \int Z(u, v) dv_2 + \zeta(u, v) du_2 + \int Z(u', v') dv_2 + \zeta(u', v') du_2 \\ &= \log \frac{\sigma(u_1 + u_2, v_1 + v_2)}{\sigma(u_1 - u_2, v_1 - v_2)} - 2v_1 Z(u_1, v_1) - 2u_1 \zeta(u_1, v_1) \dots (6), \end{aligned}$$

which may be considered a canonical form of the integral of the third kind.

In the same way integrating (2) and the relation corresponding to (4), with respect to u_1 and v_1 , I get

$$\begin{aligned} \int_{0,0} \frac{1}{\Phi} \left(\frac{\partial\Phi}{\partial v_2} dv_1 + \frac{\partial\Phi}{\partial u_2} du_1 \right) \\ = \log \frac{\sigma(u_1 + u_2, v_1 + v_2)}{\sigma(u_1 - u_2, v_1 - v_2)} - 2v_1 Z(u_2, v_2) - 2u_1 \zeta(u_2, v_2) \dots (7), \end{aligned}$$

From these two, by subtraction,

$$\begin{aligned} \int_{0,0} \frac{1}{2\Phi} \left\{ \frac{\partial\Phi}{\partial v_1} dv_2 + \frac{\partial\Phi}{\partial u_1} du_2 - \frac{\partial\Phi}{\partial v_2} dv_1 - \frac{\partial\Phi}{\partial u_2} du_1 \right\} \\ = v_1 Z(u_2, v_2) + u_1 \zeta(u_2, v_2) - v_2 Z(u_1, v_1) - u_2 \zeta(u_1, v_1) \dots (8), \end{aligned}$$

the theorem for the interchange of argument and parameter.

The function $\sigma(u, v)$.

§ 22. From

$$Z(u, v) + Z(u', v') - 2Z(u_1, v_1) = \frac{\partial}{\partial v_1} (\log \Phi),$$

$$\zeta(u, v) + \zeta(u', v') - 2\zeta(u_1, v_1) = \frac{\partial}{\partial u_1} (\log \Phi),$$

I get, on integrating with respect to u, v ,

$$\log \sigma(u, v) + \log \sigma(u', v') - 2 \log \sigma(u_1, v_1) + \text{const.} = \log \Phi.$$

That is

$$\frac{\sigma(u_1 + u_2, v_1 + v_2) \sigma(u_1 - u_2, v_1 - v_2)}{\sigma^2(u_1, v_1) \sigma^2(u_2, v_2)} = -\Phi$$

$$= \gamma_2 - \gamma_1 + \alpha_2 \beta_1 - \alpha_1 \beta_2, \dots (9);$$

for, putting u, v very small in this I get, since σ is an odd function Lt. $-\frac{1}{\sigma^2(u_1, v_1)}$ on the left-hand side, and $-\text{Lt. } \gamma_1$ on the right-hand side, and these are equal to one another.

$$\text{Since} \quad \alpha = -4 \frac{\partial^2}{\partial v^2} (\log \sigma),$$

$$\beta = 4 \frac{\partial^2}{\partial u \partial v} (\log \sigma),$$

$$\gamma = -4 \frac{\partial^2}{\partial u^2} (\log \sigma),$$

I get, by adding the half period ω to (u, v) ,

$$\alpha(u + \omega) = \frac{\gamma}{\beta} = -4 \frac{\partial^2}{\partial v^2} \{ \log \sigma(u + \omega) \},$$

$$\beta(u + \omega) = \frac{1}{\beta} = 4 \frac{\partial^2}{\partial u \partial v} \{ \log \sigma(u + \omega) \},$$

$$\gamma(u + \omega) = \frac{\alpha}{\beta} = -4 \frac{\partial^2}{\partial u^2} \{ \log \sigma(u + \omega) \}.$$

But the equations (18) of § 6 give me

$$2 \frac{\partial^2}{\partial u^2} (\log \beta) = \gamma - \frac{\alpha}{\beta} = -4 \frac{\partial^2}{\partial u^2} \left(\log \frac{\sigma(u)}{\sigma(u + \omega)} \right),$$

$$2 \frac{\partial^2}{\partial u \partial v} (\log \beta) = \frac{1}{\beta} - \beta = -4 \frac{\partial^2}{\partial u \partial v} \left(\log \frac{\sigma(u)}{\sigma(u + \omega)} \right),$$

$$2 \frac{\partial^2}{\partial v^2} (\log \beta) = \alpha - \frac{\gamma}{\beta} = -4 \frac{\partial^2}{\partial v^2} \left(\log \frac{\sigma(u)}{\sigma(u + \omega)} \right),$$

From these equations I get

$$\beta(u, v) = \left[\frac{\sigma(u + \omega, r + \omega')}{\sigma(u, v)} \right]^2 \dots (10),$$

the exactness of the relation is verified by putting $u, v = -\frac{1}{2}\omega, -\frac{1}{2}\omega'$ when both sides become unity, for I have

$$\beta(u - \omega) = 1/\beta(u),$$

and therefore

$$\{\beta(\frac{1}{2}\omega, \frac{1}{2}\omega')\}^2 = 1.$$

Now from the fundamental equation

$$\frac{\sigma(u_1 + u_2, v_1 + v_2) \sigma(u_1 - u_2, v_1 - v_2)}{\sigma^2(u_1, v_1) \sigma^2(u_2, v_2)} = -\Phi$$

$$= \gamma_2 - \gamma_1 + \alpha_2 \beta_1 - \alpha_1 \beta_2$$

I get, by adding (ω, ω') to (u_1, v_1) ,

$$\frac{\sigma(u_1 + u_2 + \omega) \sigma(u_1 - u_2 + \omega)}{\sigma^2(u_1 + \omega) \sigma^2(u_2)} = -\frac{1}{\beta_1} \Phi'$$

$$= \frac{1}{\beta_1} (\alpha_2 - \alpha_1 + \gamma_2 \beta_1 - \gamma_1 \beta_2) \dots (11),$$

from which, since, as I have just shown,

$$\beta_1 = \sigma^2(u_1 + \omega) / \sigma^2(u_1),$$

I get

$$\frac{\sigma(u_1 + u_2 + \omega) \sigma(u_1 - u_2 + \omega)}{\sigma^2(u_1) \sigma^2(u_2)} = -\Phi'$$

$$= \alpha_2 - \alpha_1 + \gamma_2 \beta_1 - \gamma_1 \beta_2 \dots (12).$$

§ 23. From these two relations we can deduce the relation for the double σ functions corresponding to what Halphen calls "l'équation à trois termes" for the elliptic σ functions.*

Putting $\sigma(u + \omega, v + \omega') \equiv \sigma_0(u, v)$,

and taking $(a, a'), (b, b'), (c, c'), (d, d')$ any four pairs of arguments, the relation is

$$\begin{aligned} & \sigma(a - b) \sigma(a + b) \sigma(c - d) \sigma(c + d) \\ & + \sigma(b - c) \sigma(b + c) \sigma(a - d) \sigma(a + d) \\ & + \sigma(c - a) \sigma(c + a) \sigma(b - d) \sigma(b + d) \\ & + \sigma_0(a - b) \sigma_0(a + b) \sigma_0(c - d) \sigma_0(c + d) \\ & + \sigma_0(b - c) \sigma_0(b + c) \sigma_0(a - d) \sigma_0(a + d) \\ & + \sigma_0(c - a) \sigma_0(c + a) \sigma_0(b - d) \sigma_0(b + d) = 0 \dots (13), \end{aligned}$$

* *Traité des fonctions elliptiques*, 1re partie, p. 187.

where, of course, $\sigma(a-b)$ is written for

$$\sigma(a-b, a'-b'),$$

and similar abbreviations are used for the other terms.

This is verified as follows:—

Writing $\Phi_{rs} = \gamma_r - \gamma_s + \alpha_r \beta_s - \alpha_s \beta_r,$

$$\Phi'_{rs} = \alpha_r - \alpha_s + \gamma_r \beta_s - \gamma_s \beta_r,$$

I shall get

$$\begin{aligned} & \Phi_{12} \Phi_{34} + \Phi_{23} \Phi_{14} + \Phi_{31} \Phi_{24} \\ &= \begin{vmatrix} \gamma_1, \alpha_1, \beta_1, 1 \\ \gamma_2, \alpha_2, \beta_2, 1 \\ \gamma_3, \alpha_3, \beta_3, 1 \\ \gamma_4, \alpha_4, \beta_4, 1 \end{vmatrix} \dots\dots\dots (14), \end{aligned}$$

for, picking out the coefficient of γ_1 on the left-hand side, I get

$$\alpha_2 \beta_4 - \alpha_4 \beta_2 + \alpha_3 \beta_2 - \alpha_3 \beta_3 + \alpha_4 \beta_3 - \alpha_4 \beta_4,$$

and, moreover, if any two of the numbers 1, 2, 3, 4 be interchanged, both sides are reproduced with changed sign.

Similarly, I shall get

$$\begin{aligned} & \Phi'_{12} \Phi'_{34} + \Phi'_{23} \Phi'_{14} + \Phi'_{31} \Phi'_{24} \\ &= \begin{vmatrix} \alpha_1, \gamma_1, \beta_1, 1 \\ \alpha_2, \gamma_2, \beta_2, 1 \\ \alpha_3, \gamma_3, \beta_3, 1 \\ \alpha_4, \gamma_4, \beta_4, 1 \end{vmatrix} \dots\dots\dots (15), \end{aligned}$$

and therefore

$$\Phi_{12} \Phi_{34} + \Phi_{23} \Phi_{14} + \Phi_{31} \Phi_{24} + \Phi'_{12} \Phi'_{34} + \Phi'_{23} \Phi'_{14} + \Phi'_{31} \Phi'_{24} = 0,$$

so that the required relation among the σ functions is at once found from (14) and (15), by taking the suffixes 1, 2, 3, 4 to correspond to the arguments (a, a') , (b, b') , (c, c') , (d, d') respectively.

III.

The Periods.

§ 24. The half-periods are, of course, given by those values of the arguments for which $\frac{\partial \alpha}{\partial v}$, $\frac{\partial \beta}{\partial v}$, $\frac{\partial \beta}{\partial u}$, &c., have zero or infinite values. One period, viz. 2ω , has been already noticed.

The equations (vi) and (vii) of § 6 are

$$\left(\frac{\partial\beta}{\partial u}\right)^2 = \beta^2\gamma + \alpha\beta + d\beta^2,$$

$$\left(\frac{\partial\beta}{\partial v}\right)^2 = \beta^2\alpha + \beta\gamma + b\beta^2.$$

From these, if $\frac{\partial\beta}{\partial u} = 0$ and $\frac{\partial\beta}{\partial v} = 0$, we get either $\beta = 0$ or

$$\left. \begin{aligned} \alpha + \gamma\beta + d\beta &= 0 \\ \gamma + \alpha\beta + b\beta &= 0 \end{aligned} \right\}.$$

(i). When $\beta = 0$.

The fundamental equation (i), viz.

$$(\alpha\gamma + 1 - c\beta + \beta^2)^2 = 4(\alpha + \beta\gamma + d\beta)(\gamma + \alpha\beta + b\beta),$$

gives, when we put $\beta = 0$,

$$(\alpha\gamma - 1)^2 = 0.$$

Therefore $\gamma = \frac{1}{\alpha}$, and from equation (ix),

$$\left(\frac{\partial\alpha}{\partial v}\right)^2 = \gamma + d + c\alpha + b\alpha^2 + \alpha^2 - \alpha\beta,$$

I get, putting $\frac{\partial\alpha}{\partial v} = 0$, $\beta = 0$, $\gamma = \frac{1}{\alpha}$,

$$\alpha^4 + b\alpha^3 + c\alpha^2 + d\alpha + 1 = 0 \dots\dots\dots(1).$$

The values of α are therefore the roots of the equation $\Theta/\theta = 0$, and I get for what I shall call the half periods $\omega_1, \omega_2, \omega_3, \omega_4$

$$\alpha(\omega_r, \omega_r') = e_r, \quad \beta(\omega_r, \omega_r') = 0, \quad \gamma(\omega_r, \omega_r') = \frac{1}{e_r}, \quad (r = 1, 2, 3, 4) \\ \dots\dots\dots(2).$$

$$(ii). \text{ When } \left. \begin{aligned} \alpha + \gamma\beta + d\beta &= 0 \\ \gamma + \alpha\beta + b\beta &= 0 \end{aligned} \right\} \dots\dots\dots(3),$$

I get at once

$$\alpha\gamma + 1 - c\beta + \beta^2 = 0$$

from (i), and therefore, eliminating α and γ ,

$$\beta^2(b\beta - d)(d\beta - b) + (1 - \beta^2)^2(1 - c\beta + \beta^2) = 0.$$

The roots of this equation are the products in pairs of the roots of the equation

$$\Theta/\theta = 0.$$

Written out in full it is

$$\beta^6 - c\beta^5 + (bd-1)\beta^4 + (2c-b^2-d^2)\beta^3 + (bd-1)\beta^2 - c\beta + 1 = 0,$$

a reciprocal equation, as it should be.

Putting then $\beta = e_1 e_2$, I get from (3)

$$\left. \begin{aligned} \alpha &= \beta \cdot \frac{d-b\beta}{\beta^2-1} = e_1 e_2 \cdot \frac{e_1 e_2 - e_3 e_4}{e_1^2 e_2^2 - 1} (e_1 + e_2) = e_1 + e_2, \\ \gamma &= \beta \cdot \frac{b-d\beta}{\beta^2-1} = e_1 e_2 \cdot \frac{e_1^2 e_2^2 - 1}{e_1^2 e_2^2 - 1} (e_3 + e_4) = \frac{1}{e_1} + \frac{1}{e_2} \end{aligned} \right\} \dots (4),$$

since $e_1 e_2 e_3 e_4 = 1$.

Expressions for $\beta(u + \omega_r, v + \omega_r')$, &c.

§ 25. I write β_1 for $\beta(u, v)$, β for $\beta(u + \omega_r, v + \omega_r')$, β' for $\beta(u - \omega_r, v - \omega_r')$, β_r for $\beta(\omega_r, \omega_r')$, with corresponding interpretations for $\alpha_1, \alpha, \alpha', \alpha_r$ and $\gamma_1, \gamma, \gamma', \gamma_r$, so as to make the notation correspond with that used in § 18 dealing with the addition theorem.

$$\text{Then} \quad \left(\frac{\partial \beta}{\partial u} \right)_r = 0, \quad \left(\frac{\partial \beta}{\partial v} \right)_r = 0,$$

$$\beta_r = 0, \quad \alpha_r = e_r, \quad \gamma_r = e_r^{-1},$$

and (v. § 14)

$$\Phi = \gamma_1 - \gamma_r + \alpha_1 \beta_r - \alpha_r \beta_1 = \gamma_1 - e_r^{-1} - e_r \beta_1,$$

$$\Phi' = \alpha_1 - \alpha_r + \gamma_1 \beta_r - \gamma_r \beta_1 = \alpha_1 - e_r - e_r^{-1} \beta_1,$$

$$\Delta = \alpha_1 \gamma_1 \beta_r - \alpha_r \gamma_r \beta_1 + (\beta_1 - \beta_r)(1 - \beta_1 \beta_r) = 0.$$

$$\text{Also} \quad \frac{\partial \Phi}{\partial v_r} = 0, \quad \frac{\partial \Phi}{\partial u_r} = 0, \quad \frac{\partial \Phi'}{\partial v_r} = 0, \quad \frac{\partial \Phi'}{\partial u_r} = 0.$$

Further

$$\begin{aligned} E &= (\beta_1 - \beta_r)^2 + (\alpha_1 - \alpha_r)(\alpha_1 \beta_r - \alpha_r \beta_1) = \beta_1 \{ \beta_1 - e_r(\alpha_1 - e_r) \} \\ &= -e_r \beta_1 \Phi'. \end{aligned}$$

From the equations

$$\alpha - \alpha' = -\frac{4}{\partial v_1 \partial v_r} (\log \Phi),$$

$$\beta - \beta' = \frac{4}{\partial v_1 \partial u_r} (\log \Phi),$$

$$\gamma - \gamma' = -\frac{4}{\partial u_1 \partial u_r} (\log \Phi),$$

I get at once $\alpha = \alpha'$, $\beta = \beta'$, $\gamma = \gamma'$, and therefore $2\omega_r$ is a period.

Also $\beta\beta_1\beta_r = \frac{D^2}{A^2}$; and since

$$A = -\frac{\partial \Phi}{\partial v_1} + \frac{\partial \Phi}{\partial v_r}, \quad A' = -\frac{\partial \Phi}{\partial v_1} - \frac{\partial \Phi}{\partial v_r},$$

$$D = \beta_1^2 \beta_r^2 \left\{ \frac{\partial}{\partial u_1} \left(\frac{\Phi'}{\beta_1 \beta_r} \right) - \frac{\partial}{\partial u_r} \left(\frac{\Phi'}{\beta_1 \beta_r} \right) \right\},$$

$$D' = \beta_1^2 \beta_r^2 \left\{ \frac{\partial}{\partial u_1} \left(\frac{\Phi'}{\beta_1 \beta_r} \right) + \frac{\partial}{\partial u_r} \left(\frac{\Phi'}{\beta_1 \beta_r} \right) \right\},$$

I get $A = A'$, but $\text{Lt. } D = -\text{Lt. } D'$, and therefore

$$\beta\beta_1\beta_r = -\frac{DD'}{AA'} = -\frac{\Phi'}{\Phi} \beta_1\beta_r,$$

that is

$$\beta = -\frac{\alpha_1 - e_r - e_r^{-1}\beta_1}{\gamma_1 - e_r^{-1} - e_r\beta_1} = \frac{e_r\alpha_1 - e_r^2 - \beta_1}{1 + e_r^2\beta_1 - e_r\gamma_1} \dots\dots\dots(5).$$

To find α , I take the formula

$$\alpha + \alpha_1 + \alpha_r = \frac{E^2}{A^2} - \frac{2B}{A},$$

then, as before, I have $A = A'$ and therefore $A^2 = E\Phi$.

$$\text{Also} \quad A\alpha_1 + B - \frac{1}{\beta_1}D + E\frac{1}{\beta_1}\left(\frac{\partial \beta}{\partial v}\right)_1 = 0.$$

I get But $\frac{D'}{A^2} = \beta_1 \beta_1 \beta_1$, and therefore, since $\beta_r = 0$, $D = 0$, and

$$\begin{aligned} \frac{B}{A} + \alpha_1 &= -\frac{E}{A^2} \cdot \frac{A}{\beta_1} \left(\frac{\partial \beta}{\partial v_1} \right)_1 = \frac{1}{\Phi} \cdot \frac{1}{\beta_1} \left(\frac{\partial \beta}{\partial v} \right)_1 \frac{\partial \Phi}{\partial v_1} \\ &= \frac{1}{\Phi} \cdot \frac{1}{\beta_1} \left(\frac{\partial \beta}{\partial v} \right)_1 \left\{ \left(\frac{\partial \gamma}{\partial v} \right)_1 - e_r \left(\frac{\partial \beta}{\partial v} \right)_1 \right\} \\ &= \frac{1}{\Phi \beta_1} \left\{ - \left(\frac{\partial \beta}{\partial v} \right)_1 \left(\frac{\partial \beta}{\partial u} \right)_1 - e_r \left(\frac{\partial \beta}{\partial v} \right)_1^2 \right\} \\ &= -\frac{1}{\Phi} \left\{ \frac{1}{2} (c\beta_1 - \alpha_1 \gamma_1 - 1 - \beta_1^2) + e_r (\gamma_1 + \alpha_1 \beta_1 + b\beta_1) \right\} \end{aligned}$$

from (vii), (viii).

Therefore substituting

$$\begin{aligned} \alpha &= -\alpha_r + \alpha_1 + \frac{1}{\Phi} (\beta_1^2 - e_r \alpha_1 \beta_1 + e_r^2 \beta_1) \\ &\quad + \frac{2}{\Phi} \left\{ \frac{1}{2} (c\beta_1 - \alpha_1 \gamma_1 - 1 - \beta_1^2) + e_r (\gamma_1 + \alpha_1 \beta_1 + b\beta_1) \right\} \\ &= \frac{\alpha_1 - \beta_1 (ce_r + 2be_r^2 + 2e_r^3) - e_r^2 \gamma_1}{1 + e_r^2 \beta_1 - e_r \gamma_1} \dots\dots\dots(6). \end{aligned}$$

By adding the half period ω to the variables (u, v) , $\alpha, \beta, \gamma, \alpha_1, \beta_1, \gamma_1$ are changed into $\frac{\gamma}{\beta}, \frac{1}{\beta}, \frac{\alpha}{\beta}, \frac{\gamma_1}{\beta_1}, \frac{1}{\beta_1}, \frac{\alpha_1}{\beta_1}$ respectively, and it is easy to deduce the expression for γ , viz.

$$\gamma = \frac{e_r^2 \alpha_1 + (ce_r + 2be_r^2 + 2e_r^3) - \gamma_1}{1 + e_r^2 \beta_1 - e_r \gamma_1} \dots\dots\dots(7).$$

In these expressions if e_r be written for e_1 , and u, v we put equal to ω_1, ω_1 so that $\alpha_1 = e_1, \beta_1 = 0, \gamma_1 = e_1^{-1}$, I get

$$\left. \begin{aligned} \beta(\omega_1 + \omega_2, \omega_1' + \omega_2') &= e_1 e_2, \\ \alpha(\omega_1 + \omega_2, \omega_1' + \omega_2') &= e_1 + e_2, \\ \gamma(\omega_1 + \omega_2, \omega_1' + \omega_2') &= \frac{1}{e_2} + \frac{1}{e_1} \end{aligned} \right\} \dots\dots\dots(8).$$

so that the six half periods of § 24 (ii) are the sums in pairs of the four half periods ω_r .

Expressions for $\beta(u + \omega_r + \omega_s, v + \omega_r' + \omega_s')$.

§ 26. As before I write β for $\beta(u + \omega_1 + \omega_2)$, β' for $\beta(u - (\omega_1 + \omega_2))$, β_1 for $\beta(u)$, &c., &c., and β_{12} for $\beta(\omega_1 + \omega_2)$, &c.

Then I shall have

$$\left(\frac{\partial \alpha}{\partial v}\right)_{12}, \quad \left(\frac{\partial \beta}{\partial v}\right)_{12}, \quad \left(\frac{\partial \beta}{\partial u}\right)_{12}, \quad \&c. \text{ all zero but } \beta_{12} \neq 0,$$

and so $A = A'$ and $D = D'$, and therefore

$$\begin{aligned} \beta &= \frac{1}{\beta_1 \beta_{12}} \frac{D''}{A''} = \frac{1}{\beta_1 \beta_{12}} \frac{DD'}{AA'} = \frac{\Phi'}{\Phi} = \frac{\alpha_1 - \alpha_{12} + \gamma_1 \beta_{12} - \gamma_{12} \beta_1}{\gamma_1 - \gamma_{12} + \alpha_1 \beta_{12} - \alpha_{12} \beta_1} \\ &= \frac{\alpha_1 - e_1 - e_2 + e_1 e_2 \gamma_1 - (e_3^{-1} + e_4^{-1}) \beta_1}{\gamma_1 - e_3^{-1} - e_4^{-1} + e_1 e_2 \alpha_1 - (e_1 + e_2) \beta_1} \\ &= \frac{\alpha_1 - (e_1 + e_2) + e_1 e_2 \gamma_1 - e_1 e_2 (e_3 + e_4) \beta_1}{\gamma_1 - e_1 e_2 (e_3 + e_4) + e_1 e_2 \alpha_1 - (e_1 + e_2) \beta_1}; \end{aligned}$$

which, since $e_1 + e_2 + e_3 + e_4 = -b$, may be written

$$\beta = \frac{\alpha_1 - (e_1 + e_2) + e_1 e_2 \gamma_1 + e_1 e_2 (b + e_1 + e_2) \beta_1}{\gamma_1 + e_1 e_2 (b + e_1 + e_2) + e_1 e_2 \alpha_1 - (e_1 + e_2) \beta_1} \dots\dots\dots (9);$$

Also from the formula

$$\begin{aligned} \alpha &= \frac{1}{\Phi^2} \left(\frac{\partial \Phi}{\partial v_1} + \frac{\partial \Phi}{\partial v_{12}} \right)^2 - \frac{2}{\Phi^2} \frac{\partial^2 \Phi}{\partial v_1 \partial v_{12}} + \frac{\beta_1 + \beta_{12}}{2\beta_1 \beta_{12}} \frac{\Delta}{\Phi} \\ &\quad - \frac{1}{\beta_1} \left(\frac{\partial \beta}{\partial v} \right)_1 \frac{1}{\Phi} \frac{\partial \Phi}{\partial v_1} - \frac{1}{\beta_{12}} \left(\frac{\partial \beta}{\partial v} \right)_{12} \frac{1}{\Phi} \frac{\partial \Phi}{\partial v_{12}}, \end{aligned}$$

since we have $\frac{\partial \Phi}{\partial v_{12}} = 0$, &c., and therefore $\left(\frac{\partial \Phi}{\partial v_1}\right)^2 = E\Phi$, and also

$$\frac{\partial \Phi}{\partial v_1} = -A = -\frac{\beta_1 - \beta_{12}}{\beta_1} \left(\frac{\partial \beta}{\partial u} \right)_1 + \frac{\alpha_1 \beta_{12} - \alpha_{12} \beta_1}{\beta_1} \left(\frac{\partial \beta}{\partial v} \right),$$

I get

$$\begin{aligned} \alpha &= \frac{E}{\Phi} + \frac{\beta_1 + \beta_{12}}{2\beta_1 \beta_{12}} \frac{\Delta}{\Phi} - \frac{1}{\beta_1 \Phi} \left\{ \frac{\alpha_1 \beta_{12} - \alpha_{12} \beta_1}{\beta_1} \left(\frac{\partial \beta}{\partial v} \right)_1^2 \right. \\ &\quad \left. - \frac{\beta_1 - \beta_{12}}{\beta_1} \left(\frac{\partial \beta}{\partial u} \right)_1 \left(\frac{\partial \beta}{\partial v} \right)_1 \right\}; \end{aligned}$$

which, on reduction, becomes

$$\alpha = \frac{e_1 e_2 (e_3 + e_4) \alpha_1 + (e_1 + e_2) \gamma_1 + e_1 e_2 (2e_1 e_2 - c) + (2e_3 e_4 - c) \beta_1}{\gamma_1 - e_1 e_2 (e_3 + e_4) + e_1 e_2 \alpha_1 - (e_1 + e_2) \beta_1},$$

which may also be written

$$\alpha = \frac{(e_1 + e_2) \gamma_1 + (2e_1^{-1} e_2^{-1} - c) \beta_1 - e_1 e_2 (b + e_1 + e_2) \alpha_1 + e_1 e_2 (2e_1 e_2 - c)}{\gamma_1 + e_1 e_2 (b + e_1 + e_2) + e_1 e_2 \alpha_1 - (e_1 + e_2) \beta_1} \dots\dots\dots(10).$$

Also by adding the half period ω to (u, v) , I deduce

$$\gamma = \frac{(e_1 + e_2) \alpha_1 + (2e_1^{-1} e_2^{-1} - c) - e_1 e_2 (b + e_1 + e_2) \gamma_1 + e_1 e_2 (2e_1 e_2 - c) \beta_1}{\gamma_1 + e_1 e_2 (b + e_1 + e_2) + e_1 e_2 \alpha_1 - (e_1 + e_2) \beta_1} \dots\dots\dots(11).$$

If in these formula (9), (10), and (11) I change e_1, e_2 into e_3, e_4 , β is changed into $1/\beta$, α into γ/β , and γ into α/β , and so since $2\omega_1, 2\omega_2, 2\omega_3$, and $2\omega_4$ are all periods, I may put

$$\omega = \omega_1 + \omega_2 + \omega_3 + \omega_4.$$

From this it follows that the fifteen distinct half periods may be arranged as follows:—

- one half period ω ,
- four half periods..... ω_r ,
- six half periods..... $\omega_r + \omega$,
- four half periods..... $\omega_r + \omega$.

IV.

The constants $\eta_1, \eta_2, \eta_3, \eta_4$, &c..

§ 27. I put

$$Z(\omega_r, \omega_r') = \eta_r',$$

$$\zeta(\omega_r, \omega_r') = \eta_r.$$

Also it is convenient to write

$$Z(\omega, \omega') = \eta',$$

$$\zeta(\omega, \omega') = \eta.$$

and it will be shown immediately that

$$\eta = \eta_1 + \eta_2 + \eta_3 + \eta_4,$$

$$\eta' = \eta_1' + \eta_2' + \eta_3' + \eta_4'.$$

Further I use $(\bar{\omega}, \bar{\omega}')$ to represent

$$(m_1\omega_1 + m_2\omega_2 + m_3\omega_3 + m_4\omega_4, \quad m_1\omega_1' + m_2\omega_2' + m_3\omega_3' + m_4\omega_4'),$$

where m represents any integer, and moreover I put

$$\bar{\eta} = m_1\eta_1 + m_2\eta_2 + m_3\eta_3 + m_4\eta_4,$$

$$\bar{\eta}' = m_1\eta_1' + m_2\eta_2' + m_3\eta_3' + m_4\eta_4'.$$

From the formula

$$Z(u_1 + u_2, v_1 + v_2) - Z(u_1 - u_2, v_1 - v_2) - 2Z(u_2, v_2) = \frac{1}{\Phi} \frac{\partial \Phi}{\partial v_2},$$

I get, on putting $(u_2, v_2) = (\omega_r, \omega_r')$,

$$Z(u_1 + \omega_r, v_1 + \omega_r) - Z(u_1 - \omega_r, v_1 - \omega_r) = 2\eta_r',$$

since $\frac{\partial \Phi}{\partial v_2} = 0$ when $u_2, v_2 = \omega_r, \omega_r'$.

Now, for brevity, writing in only one argument, this is written

$$Z(u + \omega_r) - Z(u - \omega_r) = 2\eta_r',$$

that is $Z(u + 2\omega_r) - Z(u) = 2\eta_r'$.

Similarly $Z(u + 2\omega_s) - Z(u) = 2\eta_s'$,

$$Z(u + 2\omega_s + 2\omega_r) - Z(u + 2\omega_r) = 2\eta_s',$$

and therefore

$$Z(u + 2\omega_r + 2\omega_s) - Z(u) = 2(\eta_r + \eta_s').$$

By the repeated use of these equations, I get

$$Z(u + 2\bar{\omega}) - Z(u) = 2\bar{\eta}' \dots\dots\dots(1),$$

where $\bar{\omega} = \Sigma m_r \omega_r = m_1\omega_1 + m_2\omega_2 + m_3\omega_3 + m_4\omega_4$,

$$\bar{\omega}' = \Sigma m_r \omega_r' = m_1\omega_1' + m_2\omega_2' + m_3\omega_3' + m_4\omega_4',$$

$$\bar{\eta} = \Sigma m_r \eta_r = m_1\eta_1 + m_2\eta_2 + m_3\eta_3 + m_4\eta_4,$$

$$\bar{\eta}' = \Sigma m_r \eta_r' = m_1\eta_1' + m_2\eta_2' + m_3\eta_3' + m_4\eta_4',$$

and m_r may be taken to be any integer positive or negative.

§ 28. In an exactly similar way I get

$$\zeta(u + 2\bar{\omega}) - \zeta(u) = 2\bar{\eta} \dots\dots\dots(2).$$

Now in these last two equations, writing $u - \bar{\omega}$, $v - \bar{\omega}$ for u , v , I have

$$Z(u + \bar{\omega}) - Z(u - \bar{\omega}) = 2\bar{\eta}' \dots\dots\dots(3),$$

$$\zeta(u + \bar{\omega}) - \zeta(u - \bar{\omega}) = 2\bar{\eta} \dots\dots\dots(4).$$

Multiplying these respectively by $d\bar{v}$ and du , and integrating, I get

$$\log \frac{\sigma(u + \bar{\omega})}{\sigma(\bar{\omega})} - \log \frac{\sigma(u - \bar{\omega})}{\sigma(-\bar{\omega})} = 2(\bar{\eta}u + \bar{\eta}'v),$$

or
$$-\frac{\sigma(u + \bar{\omega})}{\sigma(u - \bar{\omega})} = e^{2\bar{\eta}u + 2\bar{\eta}'v} \dots\dots\dots(5),$$

with the condition that $(\bar{\omega}, \bar{\omega}')$ must not be a period, that is that at least one of the numbers m_r is odd. This condition is necessary in order to avoid the infinities of $Z(u)$ and $\zeta(u)$.

I have then, $(\bar{\omega}, \bar{\omega}')$ being any half period,

$$\frac{\sigma(u + \bar{\omega}, v + \bar{\omega}')}{\sigma(u - \bar{\omega}, v - \bar{\omega}')} = -e^{2\bar{\eta}u + 2\bar{\eta}'v}.$$

and from this writing $u + \bar{\omega}$, $v + \bar{\omega}'$ for u , v , I get

$$\sigma(u + 2\bar{\omega}, v + 2\bar{\omega}') = -\sigma(u, v) e^{2\bar{\eta}(u + \bar{\omega}) + 2\bar{\eta}'(v + \bar{\omega}')} \dots\dots(6).$$

If $\bar{\omega}$ is a period I get by a repeated application of this formula

$$\sigma(u + 2\bar{\omega}) = \sigma(u) e^{2\bar{\eta}(u + \bar{\omega}) + 2\bar{\eta}'(v + \bar{\omega}')} \dots\dots\dots(7).$$

Both these results are included in the formula

$$\frac{\sigma(u + 2\bar{\omega}, v + 2\bar{\omega}')}{\sigma(u, v)} = -(-1)^{\Pi(m_r + 1)} e^{2\bar{\eta}(u + \bar{\omega}) + 2\bar{\eta}'(v + \bar{\omega}')} \dots\dots(8).$$

§ 29. Also, since

$$\beta(u, v) = \frac{\sigma^3(u + \bar{\omega}, v + \bar{\omega}')}{\sigma^3(u, v)},$$

and therefore

$$\frac{1}{\beta(u, v)} = \frac{\sigma^3(u + 2\bar{\omega}, v + 2\bar{\omega}')}{\sigma^3(u + \bar{\omega}, v + \bar{\omega}')},$$

I get

$$\sigma(u + 2\omega, v + 2\omega') = -\sigma(u, v),$$

and

$$\left. \begin{aligned} \eta &= \eta_1 + \eta_2 + \eta_3 + \eta_4 = 0 \\ \eta' &= \eta'_1 + \eta'_2 + \eta'_3 + \eta'_4 = 0 \end{aligned} \right\} \dots\dots\dots (9).$$

§ 30. Taking the formula

$$\frac{\sigma(u + 2\Sigma m_r \omega_r)}{\sigma(u)} = -(-1)^{\Pi(m_r+1)} \exp. 2 \{ \Sigma m_r \eta_r (u + \Sigma m_r \omega_r) + \Sigma m_r \eta'_r (v + \Sigma m_r \omega'_r) \},$$

I write $u + 2\Sigma n_r \omega_r$, $v + 2\Sigma n_r \omega'_r$ for u , v , and I get

$$\begin{aligned} \frac{\sigma(u + 2\Sigma (m_r + n_r) \omega_r)}{\sigma(u + 2\Sigma n_r \omega_r)} \\ = -(-1)^{\Pi(m_r+1)} \exp. 2 [\Sigma m_r \eta_r \{ u + \Sigma (m_r + 2n_r) \omega_r \} \\ + \Sigma m_r \eta'_r \{ v + \Sigma (m_r + 2n_r) \omega'_r \}]. \end{aligned}$$

Also

$$\frac{\sigma(u + 2\Sigma n_r \omega_r)}{\sigma(u)} = -(-1)^{\Pi(n_r+1)} \exp. 2 \{ \Sigma n_r \eta_r (u + \Sigma n_r \omega_r) + \Sigma n_r \eta'_r (v + \Sigma n_r \omega'_r) \}.$$

From these two, by multiplication,

$$\begin{aligned} \frac{\sigma\{u + 2\Sigma (m_r + n_r) \omega_r\}}{\sigma(u)} \\ = (-1)^{\Pi(m_r+1)+\Pi(n_r+1)} \exp. 2 [\Sigma (m_r + n_r) \eta_r \{ u + \Sigma (m_r + n_r) \omega_r \} \\ + \Sigma (m_r + n_r) \eta'_r \{ v + \Sigma (m_r + n_r) \omega'_r \} \\ + \Sigma m_r \eta_r \Sigma n_r \omega_r + \Sigma m_r \eta'_r \Sigma n_r \omega'_r - \Sigma n_r \eta_r \Sigma m_r \omega_r - \Sigma n_r \eta'_r \Sigma m_r \omega'_r]. \end{aligned}$$

But I have also directly from the formula

$$\begin{aligned} \frac{\sigma(u + 2\Sigma (m_r + n_r) \omega_r)}{\sigma(u)} = \\ -(-1)^{\Pi(m_r+n_r+1)} \exp. 2 [\Sigma (m_r + n_r) \eta_r \{ u + \Sigma (m_r + n_r) \omega_r \} \\ + \Sigma (m_r + n_r) \eta'_r \{ v + \Sigma (m_r + n_r) \omega'_r \}]. \end{aligned}$$

Comparing these two values for

$$\sigma\{u + 2\Sigma (m_r + n_r) \omega_r\},$$

I get

$$\text{exp. 2 } \{ \Sigma m_r \eta_r \Sigma n_r \omega_r + \Sigma m_r \eta_r' \Sigma n_r \omega_r' - \Sigma n_r \eta_r \Sigma m_r \omega_r - \Sigma n_r \eta_r' \Sigma m_r \omega_r' \}.$$

$$= -(-1) \Pi(m_r + n_r + 1) - \Pi(m_r + 1) - (n_r + 1),$$

Taking all the m 's but one zero, and likewise all the n 's but one zero, I get

$$\eta_r \omega_r + \eta_r' \omega_r' - \eta_r \omega_r - \eta_r' \omega_r' = \frac{1}{2} (2k + 1) i\pi \dots (6),$$

where k is some integer.

ON SOME PROPERTIES OF QUINTIC CURVES.

By A. B. BASSET, M.A., F.R.S.

§ 1. THE object of the following note is to show that nodal quintic curves possess various properties analogous to those of nodal quartics.

Two curves which possess the same nodes and the same nodal tangents will be called *nodotangential* curves, and we shall commence by proving the following theorem:—

If a nodal curve be cut by two nodotangential curves in two groups of ordinary points p and $p + r$ respectively, then r has a zero residual.

Let the curve C_m have a node at A , and let the nodotangential curve C_l cut C_m in a group of ordinary points of degree p . Then

$$\left. \begin{aligned} C_m &= \alpha^{m-2} u_2 + \alpha^{m-3} u_3 + \dots u_m \\ C_l &= \alpha^{l-2} u_2 + \alpha^{l-3} u_3 + \dots u_l \end{aligned} \right\} \dots \dots \dots (1),$$

also let

$$S_r = \Sigma_r^0 w_k \alpha^{r-k}, \quad S_r' = \Sigma_r^0 w_k' \alpha^{r-k} \dots \dots \dots (2),$$

and consider the curve

$$S_{l+n} = C_m S_{l+n-m}' + C_l S_n \dots \dots \dots (3),$$

where $l + n \geq m$. Multiplying out we obtain

$$S_{l+n} = \alpha^{l+n-2} (w_0' + w_0) u_2 + \dots \dots \dots (4),$$

which shows that S_{l+n} is also a nodotangential curve. Now C_m , C_l , and S_{l+n} each pass through the same cluster of six points at the node A ; also C_m and C_l intersect in a group of

ON THE KUMMER SURFACE

BY W. V. PARKER

1. *Introduction.** In his paper *On hyperelliptic functions of genus two*, A. L. Dixon† chooses as fundamental the curve, H , whose equation in the (r, s) plane may be written in the form

$$s^2 = f(r) \equiv r^5 + ar^4 + br^3 + cr^2 + r.$$

The Kummer surface associated with this curve presents several interesting features which are not found in the more usual forms. Its equation is quadratic in two of the variables. By introducing homogeneous coordinates we find that its equation possesses a three way symmetry not found in the other forms. In this form the surface has two vertices of the tetrahedron of reference as nodes and two of its faces as tropes, whereas in the usual forms we have only one of these.‡

2. *The Associated Kummer Surface.* Consider the hyperelliptic curve H of genus two in the (r, s) plane, whose equation is

$$(1) \quad s^2 = r^5 + ar^4 + br^3 + cr^2 + r,$$

where a, b, c are real and subject only to the condition that the right member does not have multiple zeros. Denote the branch points of the Riemann surface of H by $0, e_1, e_2, e_3, e_4, \infty$.

Let (α_1, ρ_1) and (α_2, ρ_2) be any two points on H , not necessarily on the real part of H . If we write

$$(2) \quad 2F(\alpha_1, \alpha_2) \equiv \alpha_1^2 \alpha_2^2 (\alpha_1 + \alpha_2) + 2a\alpha_1^2 \alpha_2^2 + b\alpha_1 \alpha_2 (\alpha_1 + \alpha_2) + 2c\alpha_1 \alpha_2 + \alpha_1 + \alpha_2,$$

equation (1) becomes $s^2 = F(r, r)$. Let us now introduce homogeneous variables (x, y, z, t) symmetrically related to these two points by the formulas

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† Quarterly Journal of Mathematics, vol. 36 (1904), p. 1.

‡ Baker, *Multiply Periodic Functions*, Cambridge University Press, 1907, Chapter 3.

$$(3) \quad \frac{x}{t} = \alpha_1 + \alpha_2, \quad \frac{y}{t} = \alpha_1 \alpha_2, \quad \frac{z}{t} = \frac{2F(\alpha_1, \alpha_2) - 2\rho_1 \rho_2}{(\alpha_1 - \alpha_2)^2}.$$

If now we eliminate α_1 and α_2 and incidentally ρ_1 and ρ_2 from equations (3), we find that x, y, z, t are connected by the relation

$$(4) \quad Y^2 - 4XZ = 0,$$

where

$$(5) \quad X = ayt + xy + zt, \quad Y = byt - y^2 - t^2 - xz, \quad Z = cyt + yz + xt.$$

The relation (4) is the equation of a Kummer surface K associated with H . The form of (4) is unaltered if we interchange x with z and y with t , a with c and x with z , or a with c and y with t .

To any two points of H there corresponds a single point of K . A point of K corresponds to two pairs of points on H which are obtained the one from the other by reflection in the r -axis. There is not, therefore, a one-to-one reciprocal correspondence between the points of K and the pairs of points of H . Such a correspondence may be established, however, by considering K as a kind of double spread. In order to distinguish between the points of the two spreads, we may introduce another pair of functions such as

$$(6) \quad \eta = \frac{\alpha_2 \rho_1 - \alpha_1 \rho_2}{\alpha_1 \alpha_2 (\alpha_1 - \alpha_2)}, \quad \zeta = \frac{\alpha_1^2 \rho_2 - \alpha_2^2 \rho_1}{\alpha_1 \alpha_2 (\alpha_1 - \alpha_2)}.$$

These functions satisfy the relations

$$yt\eta^2 = X, \quad 2yt\eta\zeta = Y, \quad yt\zeta^2 = Z,$$

where X, Y, Z are as defined by (5).

All points of K which correspond to a fixed value α_1 of r together with any other value α_i lie in the plane

$$(7) \quad \alpha_1 x - y - \alpha_1^2 t = 0.$$

For if we let $t=1$, we have $x=\alpha_1+\alpha_i$, $y=\alpha_1\alpha_i$, which substituted in (7) gives $\alpha_1(\alpha_1+\alpha_i)-\alpha_1\alpha_i-\alpha_1^2=0$, an identity in α_i . The two points of K corresponding to a given pair of values α_1 and α_2 of r are, therefore, in both the planes $\alpha_1 x - y - \alpha_1^2 t = 0$ and

$\alpha_2 x - y - \alpha_2^2 t = 0$ and hence on a line through $(0, 0, 1, 0)$. In case α_1 or α_2 is a branch point these two points of K coincide and the line from this point to $(0, 0, 1, 0)$ is tangent to K at the point, for, as will be shown presently, if α_1 is a branch point the plane (7) is a trope.

If both α_1 and α_2 are branch points, it is readily shown that the corresponding point of K is a node and the plane

$$(8) \quad \alpha_1 \alpha_2 x - (\alpha_1 + \alpha_2)y + z - 2(\alpha_1 - \alpha_2)^{-2} F(\alpha_1, \alpha_2)t = 0$$

is a trope,* that is, a plane tangent to the surface along a conic. In particular if $\alpha_1 = \alpha_2$ the node is $(0, 0, 1, 0)$ and the trope is $t = 0$, and if $\alpha_1 = 0, \alpha_2 = \infty$ the node is $(1, 0, 0, 0)$ and the trope is $y = 0$.

Since both ρ_1 and ρ_2 are zero when α_1 and α_2 are finite branch points, we see that if the sixteen nodes are (x_i, y_i, z_i, t_i) , $(i = 1, 2, \dots, 16)$, the equations of the sixteen tropes are

$$(9) \quad \begin{vmatrix} x_i & y_i \\ x & y \end{vmatrix} + \begin{vmatrix} z_i & t_i \\ z & t \end{vmatrix} = 0.$$

We establish in this way a one-to-one reciprocal correspondence between the nodes and tropes. There are, however, six nodes on each trope and six tropes on each node. We may determine how these are situated from the following table. Let the node (or trope) obtained by taking 0 with ∞ be denoted by (0), that obtained by taking 0 with e_i by (i) , that obtained by taking e_i with e_j by (i, j) , that obtained by taking ∞ with e_l by (i, j, k) and that obtained by taking any branch point with itself by $(1, 2, 3, 4)$, where i, j, k, l denote the numbers 1, 2, 3, 4 in some order. If the symbol outside denotes the node the ones inside denote the tropes on it, and if the symbol outside denotes the trope the ones inside denote the nodes on it.

(0)	[(0)	(1)	(2)	(3)	(4)	(1, 2, 3, 4)]
(i)	[(0)	(i)	(j, k)	(k, l)	(l, j)	(j, k, l)]
(i, j)	[(k)	(l)	(i, j)	(k, l)	(i, k, l)	(j, k, l)]
(i, j, k)	[(l)	(i, l)	(j, l)	(k, l)	(i, j, k)	(1, 2, 3, 4)]
(1, 2, 3, 4)	[(0)	(1, 2, 3)	(2, 3, 4)	(3, 4, 1)	(4, 1, 2)	(1, 2, 3, 4)]

* Baker, loc. cit., Chapter 3.

Equation (4) of the surface is quadratic in the variables x and z . If we write it in expanded form as a quadratic in z we have

$$\begin{aligned} & (x^2 - 4yt)z^2 - 2(xy^2 + bxyt + 2ay^2t + xt^2 + 2cyl^2)z + y^4 \\ & - 2by^3t - 4cxy^2t - 4x^2yt + (2 + b^2 - 4ac)y^2t^2 \\ & - 4axyt^2 - 2byt^3 + t^4 = 0. \end{aligned}$$

The discriminant of this quadratic in z may be written

$$\Delta_z = yt(e_1x - y - e_1^2t)(e_2x - y - e_2^2t)(e_3x - y - e_3^2t)(e_4x - y - e_4^2t).$$

Each of these linear factors equated to zero is the equation of a trope on the node $(0, 0, 1, 0)$. A similar thing is true for the discriminant if we consider the equation as a quadratic in x . In this case we get the tropes on the node $(1, 0, 0, 0)$.

3. *A Numerical Example.* The correspondence between pairs of points of H and the real points of K is not as simple as we might expect. To add to the clarity of the general discussion with particular reference to reality, let us consider a particular Kummer surface of the type just discussed. Many of the facts found here are true for the general case when e_1, e_2, e_3, e_4 are all real.

Take as the fundamental curve H ,

$$(1) \quad s^2 = r^5 - \frac{3}{2}r^4 - 2r^3 + \frac{3}{2}r^2 + r,$$

for which $e_1 = 1, e_2 = -1, e_3 = 2, e_4 = -\frac{1}{2}$. This choice of the subscripts for the e 's is absolutely arbitrary. The equation of the corresponding Kummer surface for this curve is

$$(2) \quad y^4 - 2xy^3z + x^2z^2 + (4y^3 + 4xyz - 6xy^2 - 4x^2y + 6y^2z - 4yz^2)t \\ + (15y^2 - 2xz - 6yz + 6xy)t^2 + 4yt^3 + t^4 = 0.$$

We get real points on the surface in the following cases:

1. α_1 and α_2 real with ρ_1 and ρ_2 real.
2. α_1 and α_2 real with ρ_1 and ρ_2 pure imaginary numbers.
3. α_1 and α_2 conjugate complex numbers with ρ_1 and ρ_2 conjugate complex numbers.
4. α_1 and α_2 conjugate complex numbers with ρ_2 equal to the negative of the conjugate of ρ_1 .

In order for α_1 and α_2 to change continuously from both real to conjugate complex numbers they must become equal. Points for which $\alpha_1 = \alpha_2$ lie on the cylinder

$$(3) \quad x^2 - 4yt = 0.$$

This cylinder separates the real points of the surface for which α_1 and α_2 are real from those for which α_1 and α_2 are conjugate complex numbers. Points on the real part of the surface outside of the cylinder correspond to real values of α_1 and α_2 , while those inside correspond to conjugate complex values of α_1 and α_2 .

The surfaces consists of eight *pieces*, two entirely finite and six extending to infinity. Each piece joins with four others, one at each of its nodes. Each infinite piece has two of its nodes at infinity. Two of the infinite pieces are cut by the cylinder (3) and each of these pieces is divided into six *portions* by it, three lying outside and three inside.

The real values of r are divided into six intervals by the branch points. We will indicate the intervals as follows

$$\begin{array}{lll} [1] & r \leq -1, & [3] \quad -\frac{1}{2} \leq r \leq 0, & [5] \quad 1 \leq r \leq 2, \\ [2] & r \geq 2, & [4] \quad 0 \leq r \leq 1, & [6] \quad -1 \leq r \leq -\frac{1}{2}. \end{array}$$

The end points of the intervals are regarded as belonging to both intervals. If α_1 is in $[a]$ and α_2 in $[b]$, the corresponding point of K is on a portion which we shall denote by $[a, b]$, where $[a, b]$ and $[b, a]$ are of course identical. In order to have a real point of the surface $[a]$ and $[b]$ must be both even or both odd. We may speak of the portions, therefore, as *even* and *odd* portions. The portions $[1, 3]$, $[3, 5]$, $[5, 1]$, $[2, 4]$, $[4, 6]$, $[6, 2]$ each make up one piece. They are the six pieces which are not cut by the cylinder (3). The portions $[3, 5]$ and $[4, 6]$ make up the two finite pieces. The portions $[1, 1]$, $[3, 3]$, $[5, 5]$ are on one of the pieces cut by the cylinder and the portions $[2, 2]$, $[4, 4]$, $[6, 6]$ are on the other. At each node there is one even portion and one odd portion. With the exception of the node $(0, 0, 1, 0)$ which connects the two pieces cut by the cylinder, no even portion joins with an even portion and no odd portion with an odd portion at a node.

The following table gives the sixteen nodes and tropes and also the portions joining at each node. The numbers in the first column are the symbols for the nodes (or tropes) according to the scheme given in the discussion of the general surface. The symmetry is again in evidence here. If we interchange x with z and y with t , the nodes and tropes are interchanged as follows: (0) with (1, 2, 3, 4), (i) with (j, k, l) and (i, j) with (k, l).

	nodes	tropes	portions
(0)	(1, 0, 0, 0)	$y = 0$	$[2, 4][1, 3]$
(1)	(1, 0, 1, 1)	$y - z + t = 0$	$[4, 4][3, 5]$
(2)	(1, 0, 1, -1)	$y + z + t = 0$	$[4, 6][1, 3]$
(3)	(4, 0, 1, 2)	$4y - 2z + t = 0$	$[2, 4][3, 5]$
(4)	(1, 0, 4, -2)	$y + 2z + 4t = 0$	$[4, 6][3, 3]$
(1, 2)	(0, 2, 3, -2)	$2x - 2z - 3t = 0$	$[4, 6][5, 1]$
(2, 3)	(1, -2, -1, 1)	$2x + y - z - t = 0$	$[6, 2][5, 1]$
(1, 3)	(3, 2, -3, 1)	$2x - 3y + z + 3t = 0$	$[2, 4][5, 5]$
(2, 4)	(-3, 1, 3, 2)	$x + 3y + 2z - 3t = 0$	$[6, 6][1, 3]$
(1, 4)	(1, -1, -1, 2)	$x + y - 2z - t = 0$	$[4, 6][3, 5]$
(3, 4)	(3, -2, 0, 2)	$2x + 3y - 2z = 0$	$[6, 2][3, 5]$
(1, 2, 3)	(4, -2, 1, 0)	$2x + 4y + t = 0$	$[6, 2][1, 3]$
(2, 3, 4)	(1, 1, 1, 0)	$x - y - t = 0$	$[2, 4][5, 1]$
(3, 4, 1)	(1, -1, 1, 0)	$x + y + t = 0$	$[6, 2][1, 1]$
(4, 1, 2)	(1, 2, 4, 0)	$2x - y - 4t = 0$	$[2, 2][5, 1]$
(1, 2, 3, 4)	(0, 0, 1, 0)	$t = 0$	$[2, 2][4, 4][6, 6]$ $[1, 1][3, 3][5, 5]$

ADDITION FORMULAS FOR HYPERELLIPTIC FUNCTIONS

BY W. V. PARKER

1. *Introduction.** In a recent paper the writer† discussed the Kummer surface associated with the hyperelliptic curve of the form

$$s^2 = r^5 + ar^4 + br^3 + cr^2 + r.$$

This form for the curve was used by A. L. Dixon‡ in a paper in which he obtained addition formulas for hyperelliptic functions with distinct arguments. Dixon emphasizes the unusual symmetry of this particular form for the hyperelliptic curve. In the present paper duplication formulas are found as well as the addition formulas for distinct arguments. The odd functions used here differ from those of Dixon and are slightly different from the ones ordinarily used. The particular form used here seems desirable because of the symmetry.

2. *Distinct Arguments.* Consider the fixed hyperelliptic curve, H , of genus two, given by the equation

$$(1) \quad s^2 = r^5 + ar^4 + br^3 + cr^2 + r,$$

and let L be a variable curve of the form

$$(2) \quad m_0 s = n_0 r^3 + n_1 r^2 + n_2 r + n_3, \quad m_0 \neq 0.$$

If $n_0 \neq 0$, H and L intersect in six finite points any four of which are sufficient to determine L and hence to determine the remaining two points.

Denote the six points of intersection of H and L by

$$(\alpha_1, \rho_1), (\alpha_2, \rho_2), (\beta_1, \sigma_1), (\beta_2, \sigma_2), (\gamma_1, \tau_1), (\gamma_2, \tau_2),$$

and let

* This paper is a part of a dissertation written in Brown University, 1931. The writer is indebted to A. A. Bennett for many helpful suggestions.

† This Bulletin, vol. 38 (1932), p. 403.

‡ *On hyperelliptic functions of genus two*, Quarterly Journal of Mathematics, vol. 36 (1904), p. 1.

$$\begin{aligned}
 u_1 &= \int_{\infty}^{\alpha_1} \frac{dr}{s} + \int_{\infty}^{\alpha_2} \frac{dr}{s}, & u_2 &= \int_{\infty}^{\alpha_1} \frac{r dr}{s} + \int_{\infty}^{\alpha_2} \frac{r dr}{s}, \\
 (3) \quad v_1 &= \int_{\infty}^{\beta_1} \frac{dr}{s} + \int_{\infty}^{\beta_2} \frac{dr}{s}, & v_2 &= \int_{\infty}^{\beta_1} \frac{r dr}{s} + \int_{\infty}^{\beta_2} \frac{r dr}{s}, \\
 w_1 &= \int_{\infty}^{\gamma_1} \frac{dr}{s} + \int_{\infty}^{\gamma_2} \frac{dr}{s}, & w_2 &= \int_{\infty}^{\gamma_1} \frac{r dr}{s} + \int_{\infty}^{\gamma_2} \frac{r dr}{s}.
 \end{aligned}$$

We then have by Abel's Theorem*

$$\begin{aligned}
 (4) \quad u_1 + v_1 + w_1 &\equiv 0 \pmod{\text{period}}, \\
 u_2 + v_2 + w_2 &\equiv 0 \pmod{\text{period}}.
 \end{aligned}$$

Let

$$\begin{aligned}
 \frac{x_{\alpha}}{t_{\alpha}} &= \alpha_1 + \alpha_2, & \frac{y_{\alpha}}{t_{\alpha}} &= \alpha_1 \alpha_2, \\
 \eta_{\alpha} &= \frac{\alpha_2 \rho_1 - \alpha_1 \rho_2}{\alpha_1 \alpha_2 (\alpha_1 - \alpha_2)}, & \zeta_{\alpha} &= \frac{\alpha_1^2 \rho_2 - \alpha_2^2 \rho_1}{\alpha_1 \alpha_2 (\alpha_1 - \alpha_2)};
 \end{aligned}$$

and denote similarly the corresponding expressions in $\beta_1, \beta_2, \sigma_1, \sigma_2; \gamma_1, \gamma_2, \tau_1, \tau_2$. These are all periodic functions of the integrals u_1 and u_2 . We know that x_{α}/t_{α} , and y_{α}/t_{α} are even functions† and η_{α} and ζ_{α} are odd functions. The functions $x_{\gamma}/t_{\gamma}, y_{\gamma}/t_{\gamma}, \eta_{\gamma}, \zeta_{\gamma}$ can be expressed rationally in terms of $x_{\alpha}, y_{\alpha}, t_{\alpha}, \eta_{\alpha}, \zeta_{\alpha}, x_{\beta}, y_{\beta}, t_{\beta}, \eta_{\beta}, \zeta_{\beta}$. But $x_{\gamma}/t_{\gamma}, y_{\gamma}/t_{\gamma}, \eta_{\gamma}, \zeta_{\gamma}$ are the same functions of (w_1, w_2) as $x_{\alpha}/t_{\alpha}, y_{\alpha}/t_{\alpha}, \eta_{\alpha}, \zeta_{\alpha}$ are of (u_1, u_2) and $x_{\beta}/t_{\beta}, y_{\beta}/t_{\beta}, \eta_{\beta}, \zeta_{\beta}$ are of (v_1, v_2) , and since, from (4),

$$w_1 \equiv - (u_1 + v_1),$$

and

$$w_2 \equiv - (u_2 + v_2),$$

these relations give us expressions for the functions of $(u_1 + v_1, u_2 + v_2)$ in terms of the functions of (u_1, u_2) and (v_1, v_2) , the sign being positive for even functions and negative for odd functions. Furthermore ζ_{α} can be expressed rationally in terms of $x_{\alpha}, y_{\alpha}, t_{\alpha}, \eta_{\alpha}$, and similarly η_{α} can be expressed rationally in terms of $x_{\alpha}, y_{\alpha}, t_{\alpha}, \zeta_{\alpha}$. For the functions are connected by the relations

* Appell et Goursat, *Théorie des Fonctions Algébriques*, Chap. 9.

† Dixon, loc. cit.

$$\begin{aligned}
 (5) \quad & yt\eta^2 = ay t + xy + zt, \\
 & 2yt\eta\zeta = by t - y^2 - t^2 - xz, \\
 & yt\zeta^2 = cy t + yz + xt,
 \end{aligned}$$

where

$$\frac{z}{t} = \frac{2F(\alpha_1, \alpha_2) - 2\rho_1\rho_2}{(\alpha_1 - \alpha_2)^2},$$

and

$$\begin{aligned}
 2F(\alpha_1, \alpha_2) = & \alpha_1^2\alpha_2^2(\alpha_1 + \alpha_2) + 2a\alpha_1^2\alpha_2^2 \\
 & + b\alpha_1\alpha_2(\alpha_1 + \alpha_2) + 2c\alpha_1\alpha_2 + \alpha_1 + \alpha_2.
 \end{aligned}$$

We see from these relations that interchanging a with c and y with t also interchanges η with ζ .

If we eliminate s between (1) and (2) we get the equation

$$\begin{aligned}
 (6) \quad & n_0^2 r^6 + (2n_0n_1 - m_0^2)r^5 + (n_1^2 + 2n_0n_2 - am_0^2)r^4 \\
 & + (2n_1n_2 + 2n_0n_3 - bm_0^2)r^3 \\
 & + (n_2^2 + 2n_1n_3 - cm_0^2)r^2 + (2n_2n_3 - m_0^2)r + n_3^2 = 0,
 \end{aligned}$$

whose roots are $\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2$. Hence we have

$$\begin{aligned}
 (7) \quad & \frac{x_\alpha}{t_\alpha} + \frac{x_\beta}{t_\beta} + \frac{x_\gamma}{t_\gamma} = \frac{m_0^2 - 2n_0n_1}{n_0^2}, \\
 & \frac{y_\alpha}{t_\alpha} \frac{y_\beta}{t_\beta} \frac{y_\gamma}{t_\gamma} = \frac{n_3^2}{n_0^2}.
 \end{aligned}$$

Since $(\alpha_1, \rho_1), (\alpha_2, \rho_2), (\beta_1, \sigma_1), (\beta_2, \sigma_2)$ are on L , we have

$$\begin{aligned}
 (8) \quad & m_0\rho_1 = n_0\alpha_1^3 + n_1\alpha_1^2 + n_2\alpha_1 + n_3, \\
 & m_0\rho_2 = n_0\alpha_2^3 + n_1\alpha_2^2 + n_2\alpha_2 + n_3, \\
 & m_0\sigma_1 = n_0\beta_1^3 + n_1\beta_1^2 + n_2\beta_1 + n_3, \\
 & m_0\sigma_2 = n_0\beta_2^3 + n_1\beta_2^2 + n_2\beta_2 + n_3.
 \end{aligned}$$

From (8) we get at once

$$\begin{aligned}
 (9) \quad & \eta_\alpha m_0 = \frac{x_\alpha}{t_\alpha} n_0 + n_1 - \frac{t_\alpha}{y_\alpha} n_3, \quad \zeta_\alpha m_0 = -\frac{y_\alpha}{t_\alpha} n_0 + n_2 + \frac{x_\alpha}{y_\alpha} n_3, \\
 & \eta_\beta m_0 = \frac{x_\beta}{t_\beta} n_0 + n_1 - \frac{t_\beta}{y_\beta} n_3, \quad \zeta_\beta m_0 = -\frac{y_\beta}{t_\beta} n_0 + n_2 + \frac{x_\beta}{y_\beta} n_3.
 \end{aligned}$$

If now we let $n_0 = t_\alpha t_\beta p_0$ and $n_3 = y_\alpha y_\beta p_3$ and write

$$\begin{vmatrix} x_\alpha & x_\beta \\ y_\alpha & y_\beta \end{vmatrix} = (x, y), \quad \begin{vmatrix} \eta_\alpha & 1 \\ \eta_\beta & 1 \end{vmatrix} = (\eta, 1), \text{ etc.},$$

we get

$$(10) \quad (\eta, 1)m_0 = (x, t)p_0 + (y, t)p_3, \quad (\zeta, 1)m_0 = (t, y)p_0 + (x, y)p_3.$$

Writing

$$m_0 = \begin{vmatrix} (x, t) & (y, t) \\ (t, y) & (x, y) \end{vmatrix},$$

we have

$$p_0 = \begin{vmatrix} (\eta, 1) & (y, t) \\ (\zeta, 1) & (x, y) \end{vmatrix}, \quad p_3 = \begin{vmatrix} (x, t) & (\eta, 1) \\ (t, y) & (\zeta, 1) \end{vmatrix}.$$

From (9) we get

$$(11) \quad \begin{aligned} 2n_1 &= m_0(\eta_\alpha + \eta_\beta) + p_3(y_\alpha t_\beta + y_\beta t_\alpha) - p_0(x_\alpha t_\beta + x_\beta t_\alpha), \\ 2n_2 &= m_0(\zeta_\alpha + \zeta_\beta) + p_0(y_\alpha t_\beta + y_\beta t_\alpha) - p_3(x_\alpha y_\beta + x_\beta y_\alpha) \end{aligned}$$

It is interesting to note here that interchanging y with t , and η with ζ , interchanges p_0 with p_3 , n_0 with n_3 , n_1 with n_2 , and leaves m_0 unaltered.

If we substitute the value for $2n_1$ from (11) in (7) we get

$$(12) \quad \begin{aligned} \frac{x_\gamma}{t_\gamma} &= \frac{m_0^2 - t_\alpha t_\beta p_0 [m_0(\eta_\alpha + \eta_\beta) + p_3(y_\alpha t_\beta + y_\beta t_\alpha)]}{t_\alpha^2 t_\beta^2 p_0^2}, \\ \frac{y_\gamma}{t_\gamma} &= \frac{p_3^2 y_\alpha y_\beta}{p_0^2 t_\alpha t_\beta}. \end{aligned}$$

Since the points (γ_1, τ_1) , (γ_2, τ_2) are also on L we have

$$\eta_\gamma m_0 = \frac{x_\gamma}{t_\gamma} n_0 + n_1 - \frac{t_\gamma}{y_\gamma} n_3, \quad \zeta_\gamma m_0 = -\frac{y_\gamma}{t_\gamma} n_0 + n_2 + \frac{x_\gamma}{y_\gamma} n_3.$$

If now we let $x_{\alpha+\beta}$ etc. denote the same functions of $(u_1 + v_1, u_2 + v_2)$ as x_α etc. are of (u_1, u_2) and x_β etc. are of (v_1, v_2) , we have the following addition formulas (valid for distinct arguments):

$$\begin{aligned}
 x_\gamma : y_\gamma : t_\gamma = x_{\alpha+\beta} : y_{\alpha+\beta} : t_{\alpha+\beta} = & \left\{ \begin{vmatrix} (x, t) & (y, t) \\ (t, y) & (x, y) \end{vmatrix}^2 \right. \\
 & - t_{\alpha} t_{\beta} \begin{vmatrix} (\eta, 1) & (y, t) \\ (\zeta, 1) & (x, y) \end{vmatrix} \left[(\eta_{\alpha} + \eta_{\beta}) \begin{vmatrix} (x, t) & (y, t) \\ (t, y) & (x, y) \end{vmatrix} \right. \\
 (13) \quad & \left. \left. + (y_{\alpha} t_{\beta} + y_{\beta} t_{\alpha}) \begin{vmatrix} (x, t) & (\eta, 1) \\ (t, y) & (\zeta, 1) \end{vmatrix} \right] \right\} : y_{\alpha} y_{\beta} t_{\alpha} t_{\beta} \begin{vmatrix} (x, t) & (\eta, 1) \\ (t, y) & (\zeta, 1) \end{vmatrix}^2 \\
 & : t_{\alpha}^2 t_{\beta}^2 \begin{vmatrix} (\eta, 1) & (y, t) \\ (\zeta, 1) & (x, y) \end{vmatrix}^2,
 \end{aligned}$$

$$\begin{aligned}
 \eta_\gamma = -\eta_{\alpha+\beta} &= \frac{m_0}{t_{\alpha} t_{\beta} p_0} - \frac{p_0(x_{\alpha} t_{\beta} + x_{\beta} t_{\alpha})}{2m_0} - \frac{p_0^2 t_{\alpha} t_{\beta}}{m_0 p_3} \\
 &\quad - \frac{p_3(y_{\alpha} t_{\beta} + y_{\beta} t_{\alpha})}{2m_0} - \frac{\eta_{\alpha} + \eta_{\beta}}{2}, \\
 \zeta_\gamma = -\zeta_{\alpha+\beta} &= \frac{m_0}{y_{\alpha} y_{\beta} p_3} - \frac{p_3(x_{\alpha} y_{\beta} + x_{\beta} y_{\alpha})}{2m_0} - \frac{p_3^2 y_{\alpha} y_{\beta}}{m_0 p_0} \\
 &\quad - \frac{p_0(y_{\alpha} t_{\beta} + y_{\beta} t_{\alpha})}{2m_0} - \frac{\zeta_{\alpha} + \zeta_{\beta}}{2}.
 \end{aligned}$$

These last two equations may be written

$$\begin{aligned}
 \eta_\gamma = -\eta_{\alpha+\beta} &= \frac{m_0}{t_{\alpha} t_{\beta} p_0} + \frac{p_0 t_{\alpha} t_{\beta} (\zeta, 1)}{p_3 (y, t)} - \frac{\eta_{\alpha} y_{\alpha} t_{\beta} - \eta_{\beta} y_{\beta} t_{\alpha}}{(y, t)}, \\
 \zeta_\gamma = -\zeta_{\alpha+\beta} &= \frac{m_0}{y_{\alpha} y_{\beta} p_3} + \frac{p_3 y_{\alpha} y_{\beta} (\eta, 1)}{p_0 (t, y)} - \frac{\zeta_{\alpha} y_{\beta} t_{\alpha} - \zeta_{\beta} y_{\alpha} t_{\beta}}{(t, y)}.
 \end{aligned}$$

From (12) we get

$$\frac{x_\gamma}{y_\gamma} = \frac{m_0^2 - t_{\alpha} t_{\beta} p_0 [m_0 (\eta_{\alpha} + \eta_{\beta}) + p_3 (y_{\alpha} t_{\beta} + y_{\beta} t_{\alpha})]}{y_{\alpha} y_{\beta} t_{\alpha} t_{\beta} p_3^2}.$$

If now we interchange y with t , η with ζ and p_0 with p_3 we get

$$\frac{x_\gamma}{t_\gamma} = \frac{m_0^2 - y_{\alpha} y_{\beta} p_3 [m_0 (\zeta_{\alpha} + \zeta_{\beta}) + p_0 (y_{\alpha} t_{\beta} + y_{\beta} t_{\alpha})]}{y_{\alpha} y_{\beta} t_{\alpha} t_{\beta} p_0^2}.$$

This is exactly the form obtained for x_γ/t_γ by using the coefficient of r from (6) rather than the coefficient of r^5 . A somewhat

more symmetric though much longer form for x_γ/t_γ is obtained by taking one-half the sum of these two expressions.

3. *The Coincidence Case, Duplication Formulas.* If (β_1, σ_1) coincides with (α_1, ρ_1) and also (β_2, σ_2) coincides with (α_2, ρ_2) , that is, if $u_1=v_1$ and $u_2=v_2$, the above formulas become indeterminate in the sense that they do not give the expressions for the functions of $(2u_1, 2u_2)$ in terms of the functions of (u_1, u_2) simply by setting $v_1=u_1$ and $v_2=u_2$. In order to obtain the formulas in this case as expeditiously as possible we determine the curve L so that it is tangent to H at each of the points (α_1, ρ_1) and (α_2, ρ_2) . We then have by Abel's theorem

$$2u_1 + w_1 \equiv 0 \pmod{\text{period}}, \quad 2u_2 + w_2 \equiv 0 \pmod{\text{period}}.$$

Since the roots of (6) are now $\alpha_1, \alpha_1, \alpha_2, \alpha_2, \gamma_1, \gamma_2$, we have

$$(14) \quad \frac{x_\gamma}{t_\gamma} = \frac{m_0^2 - 2n_0n_1}{n_0^2} - \frac{2x_\alpha}{t_\alpha}, \quad \frac{y_\gamma}{t_\gamma} = \frac{n_3^2 t_\alpha^2}{n_0^2 y_\alpha^2},$$

where the ratios $n_0/m_0, n_1/m_0, n_2/m_0, n_3/m_0$ are to be determined from the equations*

$$\begin{aligned} (15) \quad y_1 \eta m_0 &= x y n_0 + y l n_1 - t^2 n_3, \quad y l \zeta m_0 = -y^2 n_0 + y l n_2 + x t n_3, \\ (5xy^2 + 4ay^2t - 2cyl^2 - xt^2)m_0 &= 6(y^2t\zeta + xy^2\eta)n_0 + 4y^2t\eta n_1 - 2yt^2\zeta n_2, \\ (5y^3 - 3by^2t - 2cxyt - x^2t + yt^2)m_0 &= 6y^3\eta n_0 - 4y^2t\zeta n_1 - 2(xyt\zeta + y^2t\eta)n_2. \end{aligned}$$

The computation here is greatly simplified by introducing the function z/t as defined in (5) which is expressed rationally in terms of $x/t, y/t, \eta, \zeta$ through the relations given there. Making use of these relations and writing

$$(16) \quad \begin{aligned} p_0 &= (y^2 - t^2)\eta + (xy - zt)\zeta, & p_2 &= t\eta^2 + z\eta\zeta + y\zeta^2, \\ p_1 &= y\eta^2 + x\eta\zeta + t\zeta^2, & p_3 &= (xt - yz)\eta + (t^2 - y^2)\zeta, \end{aligned}$$

we get immediately

$$(17) \quad \begin{aligned} m_0 &= 2yt p_1, & n_0 &= t p_0, & n_1 &= t p_3 - x p_0 + 2yt p_1 \eta, \\ n_2 &= y p_0 - x p_3 + 2yt p_1 \zeta, & n_3 &= y p_3. \end{aligned}$$

* Since there is no possibility of confusion the subscript α will be omitted in what is to follow.

If now $x_{2\alpha}$ etc. denote the same functions of $(2u_1, 2u_2)$ as x_α etc. are of (u_1, u_2) , and we substitute the expressions from (17) in (14), we get the following duplication formulas:

$$x_\gamma : y_\gamma : t_\gamma = x_{2\alpha} : y_{2\alpha} : t_{2\alpha} = (4yt p_1 p_2 - 2p_0 p_3) : p_3^2 : p_0^2.$$

Since the points (γ_1, τ_1) and (γ_2, τ_2) are on L , we have

$$\eta_\gamma m_0 = \frac{x_\gamma}{t_\gamma} n_0 + n_1 - \frac{t_\gamma}{y_\gamma} n_3, \quad \zeta_\gamma m_0 = -\frac{y_\gamma}{t_\gamma} n_0 + n_2 + \frac{x_\gamma}{y_\gamma} n_3.$$

Hence we have

$$\eta_\gamma = -\eta_{2\alpha} = \frac{[(4yt p_1 p_2 - 2p_0 p_3)t - x p_0^2] p_3 + p_0(t p_3^2 - y p_0^2)}{2yt p_0 p_1 p_3} + \eta,$$

$$\zeta_\gamma = -\zeta_{2\alpha} = \frac{[(4yt p_1 p_2 - 2p_0 p_3)y - x p_3^2] p_0 + p_3(y p_0^2 - t p_3^2)}{2yt p_0 p_1 p_3} + \zeta,$$

where p_0, p_1, p_2, p_3 are as defined in (16).

For certain choices of (α_1, ρ_1) and (α_2, ρ_2) in the coincidence case, (γ_1, τ_1) will coincide with (γ_2, τ_2) and the curve L will be tangent to H at each of three points. In fact (α_1, ρ_1) can be chosen arbitrarily subject to the condition that $\rho_1 \neq 0$ and then (α_2, ρ_2) can be determined in a finite number of ways so that the curve L which is tangent to H at (α_1, ρ_1) and (α_2, ρ_2) will also be tangent at a third point. If we take a fixed point on H and determine a curve L through this point and any three of the points where H meets the r -axis, this curve L will meet H in another pair of points such that the L which is tangent to H at each of these is also tangent at the given fixed point. Furthermore all such tangent curves may be obtained in this way.*

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* For proof of this statement see a paper by the writer in this Bulletin, vol. 37 (1931), p. 557.